CS 361, Lecture 10

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Outline

- Annihilators

Annihilator Operators

We define three basic operations we can perform on this sequence:

1. Multiply the sequence by a constant: $cA = \langle ca_0, ca_1, ca_2, \cdots \rangle$
2. Shift the sequence to the left: $\mathbf{L}A = \langle a_1, a_2, a_3, \cdots \rangle$
3. Add two sequences: if $A = \langle a_0, a_1, a_2, \cdots \rangle$ and $B = \langle b_0, b_1, b_2, \cdots \rangle$, then $A + B = \langle a_0 + b_0, a_1 + b_1, a_2 + b_2, \cdots \rangle$

Annihilator Description

- We first express our recurrence as a sequence T
- *We use these three operators to "annihilate" T, i.e. make it all 0's*
- Key rule: can't multiply by the constant 0
- We can then determine the solution to the recurrence from the sequence of operations performed to annihilate T

Example

- Consider the recurrence $T(n) = 2T(n-1)$, $T(0) = 1$
- If we solve for the first few terms of this sequence, we can see they are $\langle 2^0, 2^1, 2^2, 2^3, \cdots \rangle$
- Thus this recurrence becomes the sequence:

$$T = \langle 2^0, 2^1, 2^2, 2^3, \cdots \rangle$$

Example (II)

Let's annihilate $T = \langle 2^0, 2^1, 2^2, 2^3, \cdots \rangle$

- Multiplying by a constant $c = 2$ gets:

$$2T = \langle 2 * 2^0, 2 * 2^1, 2 * 2^2, 2 * 2^3, \cdots \rangle = \langle 2^1, 2^2, 2^3, 2^4, \cdots \rangle$$

- Shifting one place to the left gets $\mathbf{L}T = \langle 2^1, 2^2, 2^3, 2^4, \cdots \rangle$
- Adding the sequence $\mathbf{L}T$ and $-2T$ gives:

$$\mathbf{L}T - 2T = \langle 2^1 - 2^1, 2^2 - 2^2, 2^3 - 2^3, \cdots \rangle = \langle 0, 0, 0, \cdots \rangle$$

Distributive Property

- The distributive property holds for these three operators
- Thus can rewrite $\mathbf{L}T - 2T$ as $(\mathbf{L} - 2)T$
- The operator $(\mathbf{L} - 2)$ annihilates T (makes it the sequence of all 0's)
- Thus $(\mathbf{L} - 2)$ is called the *annihilator* of T

0, the "Forbidden Annihilator"

- Multiplication by 0 will annihilate *any* sequence
- Thus we disallow multiplication by 0 as an operation
- In particular, we disallow $(c-c) = 0$ for any c as an annihilator
- Must always have at least one **L** operator in any annihilator!

Uniqueness

- An annihilator annihilates exactly *one* type of sequence
- In general, the annihilator $\mathbf{L} - c$ annihilates any sequence of the form $\langle a_0 c^n \rangle$
- If we find the annihilator, we can find the type of sequence, and thus solve the recurrence
- We will need to use the base case for the recurrence to solve for the constant a_0

Example

If we apply operator $(\mathbf{L} - 3)$ to sequence T above, it fails to annihilate T

$$\begin{aligned}
(\mathbf{L} - 3)T &= \mathbf{L}T + (-3)T \\
&= \langle 2^1, 2^2, 2^3, \cdots \rangle + \langle -3 \times 2^0, -3 \times 2^1, -3 \times 2^2, \cdots \rangle \\
&= \langle (2-3) \times 2^0, (2-3) \times 2^1, (2-3) \times 2^2, \cdots \rangle \\
&= (2-3)T = -T
\end{aligned}$$

Example (II)

What does $(\mathbf{L} - c)$ do to other sequences $A = \langle a_0 d^n \rangle$ when $d \neq c$?:

$$\begin{aligned}
(\mathbf{L} - c)A &= (\mathbf{L} - c)\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle \\
&= \mathbf{L}\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle - c\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle \\
&= \langle a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle - \langle c a_0, c a_0 d, c a_0 d^2, c a_0 d^3, \cdots \rangle \\
&= \langle a_0 d - c a_0, a_0 d^2 - c a_0 d, a_0 d^3 - c a_0 d^2, \cdots \rangle \\
&= \langle (d-c) a_0, (d-c) a_0 d, (d-c) a_0 d^2, \cdots \rangle \\
&= (d-c)\langle a_0, a_0 d, a_0 d^2, \cdots \rangle \\
&= (d-c)A
\end{aligned}$$

Uniqueness

- The last example implies that an annihilator annihilates one type of sequence, but does not annihilate other types of sequences
- Thus Annihilators can help us classify sequences, and thereby solve recurrences

Lookup Table

- The annihilator $\mathbf{L} - a$ annihilates any sequence of the form $\langle c_1 a^n \rangle$

Example

First calculate the annihilator:

- Recurrence: $T(n) = 4 * T(n-1)$, $T(0) = 2$
- Sequence: $T = \langle 2, 2*4, 2*4^2, 2*4^3, \cdots \rangle$
- Calulate the annihilator:
  - $\mathbf{L}T = \langle 2*4, 2*4^2, 2*4^3, 2*4^4, \cdots \rangle$
  - $4T = \langle 2*4, 2*4^2, 2*4^3, 2*4^4, \cdots \rangle$
  - Thus $\mathbf{L}T - 4T = \langle 0, 0, 0, \cdots \rangle$
  - And so $\mathbf{L} - 4$ is the annihilator

Example (II)

Now use the annihilator to solve the recurrence

- Look up the annihilator in the "Lookup Table"
- It says: "The annihilator $\mathbf{L} - 4$ annihilates any sequence of the form $\langle c_1 4^n \rangle$"
- Thus $T(n) = c_1 4^n$, but what is c_1?
- We know $T(0) = 2$, so $T(0) = c_1 4^0 = 2$ and so $c_1 = 2$
- Thus $T(n) = 2 * 4^n$

In Class Exercise

Consider the recurrence $T(n) = 3 * T(n-1)$, $T(0) = 3$,

- Q1: Calculate $T(0)$,$T(1)$,$T(2)$ and $T(3)$ and write out the sequence T
- Q2: Calculate $\mathbf{L}T$, and use it to compute the annihilator of T
- Q3: Look up this annihilator in the lookup table to get the general solution of the recurrence for $T(n)$
- Q4: Now use the base case $T(0) = 3$ to solve for the constants in the general solution

Multiple Operators

- We can apply multiple operators to a sequence
- For example, we can multiply by the constant c and then by the constant d to get the operator cd
- We can also multiply by c and then shift left to get $c\mathbf{L}T$ which is the same as $\mathbf{L}cT$
- We can also shift the sequence twice to the left to get $\mathbf{LL}T$ which we'll write in shorthand as $\mathbf{L}^2 T$

Multiple Operators

- We can string operators together to annihilate more complicated sequences
- Consider: $T = \langle 2^0 + 3^0, 2^1 + 3^1, 2^2 + 3^2, \cdots \rangle$
- We know that $(\mathbf{L} - 2)$ annihilates the powers of 2 while leaving the powers of 3 essentially untouched
- Similarly, $(\mathbf{L} - 3)$ annihilates the powers of 3 while leaving the powers of 2 essentially untouched
- Thus if we apply both operators, we'll see that $(\mathbf{L} - 2)(\mathbf{L} - 3)$ annihilates the sequence T

The Details

- Consider: $T = \langle a^0 + b^0, a^1 + b^1, a^2 + b^2, \cdots \rangle$
- $\mathbf{L}T = \langle a^1 + b^1, a^2 + b^2, a^3 + b^3, \cdots \rangle$
- $aT = \langle a^1 + a * b^0, a^2 + a * b^1, a^3 + a * b^2, \cdots \rangle$
- $\mathbf{L}T - aT = \langle (b-a)b^0, (b-a)b^1, (b-a)b^2, \cdots \rangle$
- We know that $(\mathbf{L} - a)T$ annihilates the a terms and multiplies the b terms by $b - a$
- Thus $(\mathbf{L} - a)T = \langle (b-a)b^0, (b-a)b^1, (b-a)b^2, \cdots \rangle$
- And so the sequence $(\mathbf{L} - a)T$ is annihilated by $(\mathbf{L} - b)$
- Thus the annihilator of T is $(\mathbf{L} - b)(\mathbf{L} - a)$

Key Point

- In general, the annihilator $(\mathbf{L} - a)(\mathbf{L} - b)$ (where $a \neq b$) will anihilate *only* all sequences of the form $\langle c_1 a^n + c_2 b^n \rangle$
- We will often multiply out $(\mathbf{L} - a)(\mathbf{L} - b)$ to $\mathbf{L}^2 - (a+b)\mathbf{L} + ab$
- Left as an exercise to show that $(\mathbf{L} - a)(\mathbf{L} - b)T$ is the same as $(\mathbf{L}^2 - (a+b)\mathbf{L} + ab)T$

Lookup Table

- The annihilator $\mathbf{L} - a$ annihilates sequences of the form $\langle c_1 a^n \rangle$
- The annihilator $(\mathbf{L} - a)(\mathbf{L} - b)$ (where $a \neq b$) anihilates sequences of the form $\langle c_1 a^n + c_2 b^n \rangle$

Fibonnaci Sequence

- We now know enough to solve the Fibonnaci sequence
- Recall the Fibonnaci recurrence is $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- Let T_n be the n-th element in the sequence
- Then we've got:

$$T = \langle T_0, T_1, T_2, T_3, \cdots \rangle \quad (1)$$
$$\mathbf{L}T = \langle T_1, T_2, T_3, T_4, \cdots \rangle \quad (2)$$
$$\mathbf{L}^2 T = \langle T_2, T_3, T_4, T_5, \cdots \rangle \quad (3)$$

- Thus $\mathbf{L}^2 T - \mathbf{L}T - T = \langle 0, 0, 0, \cdots \rangle$
- In other words, $\mathbf{L}^2 - \mathbf{L} - 1$ is an annihilator for T

Factoring

- $\mathbf{L}^2 - \mathbf{L} - 1$ is an annihilator that is not in our lookup table
- However, we can *factor* this annihilator (using the quadratic formula) to get something similar to what's in the lookup table
- $\mathbf{L}^2 - \mathbf{L} - 1 = (\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Quadratic Equation

"Me fail English? That's Unpossible!" - Ralph, the Simpsons

High School Algebra Review:

- To factor something of the form $ax^2 + bx + c$, we use the *Quadratic Formula*:
- $ax^2 + bx + c$ factors into $(x - \phi)(x - \hat{\phi})$, where:

$$\phi = \frac{b + \sqrt{b^2 - 4ac}}{2a} \quad (4)$$
$$\hat{\phi} = \frac{b - \sqrt{b^2 - 4ac}}{2a} \quad (5)$$

Example

- To factor: $\mathbf{L}^2 - \mathbf{L} - 1$
- Rewrite: $1 * \mathbf{L}^2 - 1 * \mathbf{L} - 1$, $a = 1$, $b = -1$, $c = -1$
- From Quadratic Formula: $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- So $\mathbf{L}^2 - \mathbf{L} - 1$ factors to $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$

Back to Fibonnaci

- Recall the Fibonnaci recurrence is $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- We've shown the annihilator for T is $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- If we look this up in the "Lookup Table", we see that the sequence T must be of the form $\langle c_1 \phi^n + c_2 \hat{\phi}^n \rangle$
- All we have left to do is solve for the constants c_1 and c_2
- Can use the base cases to solve for these

Finding the Constants

- We know $T = \langle c_1 \phi^n + c_2 \hat{\phi}^n \rangle$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- We know

$$
\begin{aligned}
T(0) &= c_1 + c_2 = 0 \quad (6) \\
T(1) &= c_1 \phi + c_2 \hat{\phi} = 1 \quad (7)
\end{aligned}
$$

- We've got two equations and two unknowns
- Can solve to get $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$,

The Punchline

- Recall Fibonnaci recurrence: $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- The final explicit formula for $T(n)$ is thus:

$$T(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

(Amazingly, $T(n)$ is *always* an integer, in spite of all of the square roots in its formula.)

Todo

- Finish hw2!