CS 461, Lecture 15

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Today's Outline

- Data Structures for Disjoint Sets

Disjoint Sets

- A disjoint set data structure maintains a collection $\{S_1, S_2, \ldots S_k\}$ of disjoint dynamic sets
- Each set is identified by a representative which is a member of that set
- Let's call the members of the sets *objects*.

Operations

We want to support the following operations:

- Make-Set(x): creates a new set whose only member (and representative) is x
- Union(x,y): unites the sets that contain x and y (call them S_x and S_y) into a new set that is $S_x \cup S_y$. The new set is added to the data structure while S_x and S_y are deleted. The representative of the new set is any member of the set.
- Find-Set(x): Returns a pointer to the representative of the (unique) set containing x

Simple Union

```
Make-Set(x){
  parent(x) = x;
  size(x) = 1;
}
Simple-Union(x,y){
  xRep = Find-Set(x);
  yRep = Find-Set(y);
  if (size(xRep)) > size(yRep)){
    parent(yRep) = xRep;
  }else{
    parent(xRep) = yRep;
  }
  size(yRep) = size(yRep) + size(xRep);
}
```


Analysis

- We showed in last class that the heights of all trees are no more than logarithmic in the number of nodes in the tree
- Thus all of these operations take $O(\log n)$ time
- Q: Can we do better?
- A: Yes we can do much better in an amortized sense.

Shallow Threaded Trees

- One good idea is to just have every object keep a pointer to the leader of it's set
- In other words, each set is represented by a tree of depth 1
- Then Make-Set and Find-Set are completely trivial, and they both take $O(1)$ time
- Q: What about the Union operation?

Union

- To do a union, we need to set all the leader pointers of one set to point to the leader of the other set
- To do this, we need a way to visit all the nodes in one of the sets
- We can do this easily by "threading" a linked list through each set starting with the sets leaders
- The threads of two sets can be merged by the Union algorithm in constant time

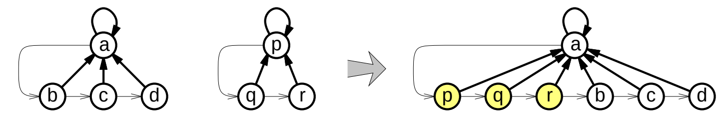
The Code

```
Make-Set(x){
  leader(x) = x;
  next(x) = NULL;
}
Find-Set(x){
  return leader(x);
}
```


The Code

```
Union(x,y){
  xRep = Find-Set(x);
  yRep = Find-Set(y);
  leader(y) = xRep;
  while(next(y)!=NULL){
    y = next(y);
    leader(y) = xRep;
  }
  next(y) = next(xRep);
  next(xRep) = yRep;
}
```


Example



Merging two sets stored as threaded trees.
Bold arrows point to leaders; lighter arrows form the threads.
Shaded nodes have a new leader.

Analysis

- Worst case time of Union is a constant times the size of the *larger* set
- So if we merge a one-element set with a n element set, the run time can be $\Theta(n)$
- In the worst case, it's easy to see that n operations can take $\Theta(n^2)$ time for this alg

Problem

- The main problem here is that in the worst case, we always get unlucky and choose to update the leader pointers of the larger set
- Instead let's purposefully choose to update the leader pointers of the smaller set
- To do this, we will need to keep track of the sizes of all the sets

The Code

```
Make-Weighted-Set(x){
  leader(x) = x;
  next(x) = NULL;
  size(x) = 1;
}
```


The Code

```
Weighted-Union(x,y){
  xRep = Find-Set(x);
  yRep = Find-Set(y)
  if(size(xRep)>size(yRep){
    Union(xRep,yRep);
    size(xRep) = size(xRep) + size(yRep);
  }else{
    Union(yRep,xRep);
    size(yRep) = size(xRep) + size(yRep);
  }
}
```


Analysis

- The Weighted-Union algorithm still takes $\Theta(n)$ time to merge two n element sets
- However in an amortized sense, it is more efficient
- Intuitively, in order to merge two large sets, we need to perform a large number of cheap Weighted-Unions
- We will show that a sequence of n Make-Weighted-Set operations and m Weighted-Union operations takes $O(m+n\log n)$ time in the worst case.

Proof

- Whenever the leader of an object x is changed by a call to Weighted-Union, the size of the set containing x increases by a factor of at least 2
- Thus if the leader of x has changed k times, the set containing x has at least 2^k members
- After the sequence of operations ends, the largest set has at most n members
- Thus the leader of any object x has changed at most $\lfloor \log n \rfloor$ times

Proof

- Let n be the number of calls to Make-Weighted-Set and m be the number of calls to Weighted-Union
- Since each call to Weighted-Union reduces the number of sets by one, there are $n - m$ sets at the end of the sequence
- Further at most m objects are *not* in singleton sets
- We've shown that each of the objects that are not in singleton sets had at most $O(\log n)$ leader changes
- Thus, the total amount of work done in updating the leader pointers is $O(n \log n)$

Proof

- We've just shown that for n calls to Make-Weighted-Set and m calls to Weighted-Union, that total cost for updating leader pointers is $O(n \log n)$
- We know that other than the work needed to update these leader pointers, each call to one of our functions does only constant work
- Thus total amount of work is $O(n \log n + m)$
- Thus each Weighted-Union call has amortized cost of $O(\log n)$

Side Note: We've just used the aggregate method of amortized analysis

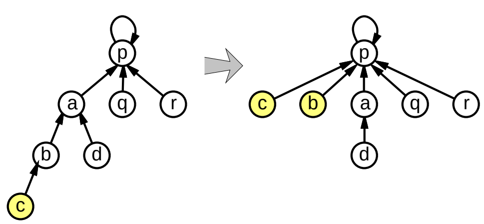
Analysis

- Using Simple-Union, *Find* takes logarithmic worst case time and everything else is constant
- Using Weighted-Union, *Union* takes logarithmic amortized time and everything else is constant
- A third method allows us to get both of these operations in *almost* constant amortized time

Path Compression

- We start with the unthreaded tree representation (from Simple-Union)
- Key Observation is that in any *Find* operation, once we get the leader of an object x, we can speed up future Find's by redirecting x's parent pointer directly to that leader
- We can also change the parent pointers of all ancestors of x all the way up to the root (We'll do this using recursion)
- This modification to Find is called path compression

Example



Path compression during Find(c). Shaded nodes have a new parent.

PC-Find Code

```
PC-Find(x){
  if(x!=Parent(x)){
    Parent(x) = PC-Find(Parent(x));
  }
  return Parent(x);
}
```


Rank

- For ease of analysis, instead of keeping track of the size of each of the trees, we will keep track of the *rank*
- Each node will have an associated rank
- This rank will give an estimate of the log of the number of elements in the set

Code

```
PC-MakeSet(x){
  parent(x) = x;
  rank(x) = 0;
}
PC-Union(x,y){
  xRep = PC-Find(x);
  yRep = PC-Find(y);
  if(rank(xRep) > rank(yRep))
    parent(yRep) = xRep;
  else{
    parent(xRep) = yRep;
    if(rank(xRep)==rank(yRep))
      rank(yRep)++;
    }
}
```


Rank Facts

- If an object x is not the set leader, then the rank of x is strictly less than the rank of its parent
- For a set X, size$(X) \geq 2^{rank(leader(X))}$ (can show using induction)
- Since there are n objects, the highest possible rank is $O(\log n)$
- Only set leaders can change their rank

Rank Facts

Can also say that there are at most $n/2^r$ objects with rank r.

- When the rank of a set leader x changes from $r-1$ to r, mark all nodes in that set. At least 2^r nodes are marked and each of these marked nodes will always have rank less than r
- There are n nodes total and any object with rank r marks 2^r of them
- Thus there can be at most $n/2^r$ objects of rank r

Blocks

- We will also partition the objects into several numbered blocks
- x is assigned to block number $\log^*(rank(x))$
- Intuitively, $\log^* n$ is the number of times you need to hit the log button on your calculator, after entering n, before you get 1
- In other words x is in block b if

$$2 \uparrow\uparrow (b-1) < rank(x) \leq 2 \uparrow\uparrow b,$$

where $\uparrow\uparrow$ is defined as in the next slide

Definition

- $2 \uparrow\uparrow b$ is the *tower* function

$$2 \uparrow\uparrow b = \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}b = \begin{cases} 1 & \text{if } b = 0 \\ 2^{2\uparrow\uparrow(b-1)} & \text{if } b > 0 \end{cases}$$

Number of Blocks

- Every object has a rank between 0 and $\lfloor \log n \rfloor$
- So the blocks numbers range from 0 to $\log^* \lfloor \log n \rfloor = \log^*(n) - 1$
- Hence there are $\log^* n$ blocks

Number Objects in Block b

- Since there are at most $n/2^r$ objects with any rank r, the total number of objects in block b is at most

$$\sum_{r=2\uparrow\uparrow(b-1)+1}^{2\uparrow\uparrow b} \frac{n}{2^r} < \sum_{r=2\uparrow\uparrow(b-1)+1}^{\infty} \frac{n}{2^r} = \frac{n}{2^{2\uparrow\uparrow(b-1)}} = \frac{n}{2 \uparrow\uparrow b}.$$

Theorem

- **Theorem:** If we use both PC-Find and PC-Union (i.e. Path Compression and Weighted Union), the worst-case running time of a sequence of m operations, n of which are MakeSet operations, is $O(m \log^* n)$
- Each PC-MakeSet aand PC-Union operation takes constant time, so we need only show that any sequence of m PC-Find operations require $O(m \log^* n)$ time in the worst case
- We will use a kind of accounting method to show this

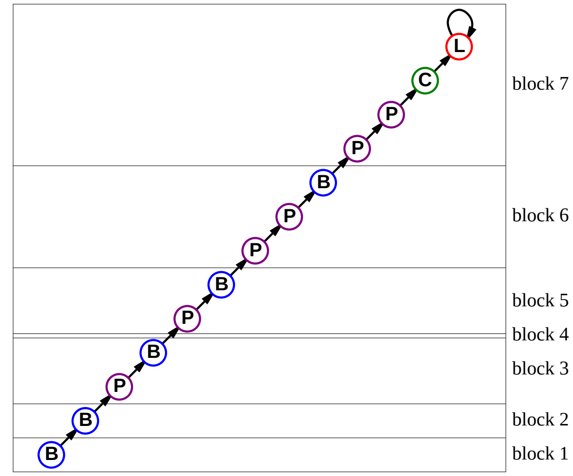
Proof

- The cost of PC-Find(x_0) is proportional to the number of nodes on the path from x_0 up to its leader
- Each object $x_0, x_1, x_2, \ldots, x_l$ on the path from x_0 to its leader will pay a 1 tax into one of several bank accounts
- After all the Find operations are done, the total amount of money in these accounts will give us the total running time

Taxation

- The leader x_l pays into the *leader* account.
- The child of the leader x_{l-1} pays into the *child* account.
- Any other object x_i in a different block from its parent x_{i+1} pays into the *block* account.
- Any other object x_i in the same block as its parent x_{i+1} pays into the *path* account.

Example



Different nodes on the find path pay into different accounts: B=block, P=path, C=child, L=leader.
Horizontal lines are boundaries between blocks. Only the nodes on the find path are shown.

Leader, Child and Block accounts

- During any Find operation, one dollar is paid into the leader account
- At most one dollar is paid into the child account
- At most one dollar is paid into the block account for each of the $\log^* n$ blocks
- Thus when the sequence of m operations ends, these accounts share a total of at most $2m + m \log^* n$ dollars

Path Account

- The only remaining difficulty is the Path account
- Consider an object x_i in block b that pays into the path account
- This object is not a set leader so its rank can never change.
- The parent of x_i is also not a set leader, so after path compression, x_i gets a new parent, x_l, whose rank is strictly larger than its old parent x_{i+1}
- Since $rank(parent(x))$ is always increasing, parent of x_i must eventually be in a different block than x_i, after which x_i will never pay into the path account
- *Thus x_i pays into the path account at most once for every rank in block b, or less than $2 \uparrow\uparrow b$ times total*

Path Account

- Since block b contains less than $n/(2 \uparrow\uparrow b)$ objects, and each of these objects contributes less than $2 \uparrow\uparrow b$ dollars, the total number of dollars contributed by objects in block b is less than n dollars to the path account
- There are $\log^* n$ blocks so the path account receives less than $n \log^* n$ dollars total
- Thus the total amount of money in all four accounts is less than $2m + m \lg^* n + n \lg^* n = O(m \lg^* n)$, and this bounds the total running time of the m operations.

Take Away

- We can now say that each call to PC-Find has amortized cost $O(\log^* n)$, which is significantly better than the worst case cost of $O(\log n)$
- The book shows that PC-Find has amortized cost of $O(A(n))$ where $A(n)$ is an even slower growing function than $\log^* n$