The Forgiving Graph: A distributed data structure for low stretch under adversarial attack

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Abstract

We consider the problem of self-healing in peer-to-peer networks that are under repeated attack by an omniscient adversary. We assume that, over a sequence of rounds, an adversary either inserts a node with arbitrary connections or deletes an arbitrary node from the network. The network responds to each such change by quick "repairs," which consist of adding or deleting a small number of edges.

These repairs essentially preserve closeness of nodes after adversarial deletions, without increasing node degrees by too much, in the following sense. At any point in the algorithm, nodes v and w whose distance would have been ℓ in the graph formed by considering only the adversarial insertions (not the adversarial deletions), will be at distance at most $\ell \log n$ in the actual graph, where n is the total number of vertices seen so far. Similarly, at any point, a node v whose degree would have been d in the graph with adversarial insertions only, will have degree at most $3d$ in the actual graph. Our algorithm is completely distributed and has low latency and bandwidth requirements.

1 Introduction

Many modern networks are *reconfigurable*, in the sense that the topology of the network can be changed by the nodes in the network. For example, peer-to-peer, wireless and mobile networks are reconfigurable. More generally, many social networks, such as a company's organizational chart; infrastructure networks, such as an airline's transportation network; and biological networks, such as the human brain, are also reconfigurable. Reconfigurable networks offer the promise of "self-healing" in the sense that when nodes in the network fail, the remaining nodes can reconfigure their links to overcome this failure. In this paper, we describe a distributed data structure for maintaining invariants in a reconfigurable network. We note that our approach is *responsive* in the sense that it responds to an attack by changing the network topology. Thus, it is orthogonal and complementary to traditional non-responsive techniques for ensuring network robustness.

This paper builds significantly on results achieved in [7], which presented a responsive, distributed data structure called the *Forgiving Tree* for maintaining a reconfigurable network in the fact of attack. The Forgiving Tree ensured two invariants: 1) the diameter of the network never increased by more than a multiplicative factor of $O(\log \Delta)$ where Δ is the maximum degree in the graph; and 2) the degree of a node never increased by more than an additive factor of 3.

In this paper, we present a new, improved distributed data structure called the *Forgiving Graph*. The improvements of the Forgiving Graph over the Forgiving Tree are threefold. First, the Forgiving Graph maintains low stretch i.e. it ensures that the distance between any pair of nodes v and w is close to what their distance would be even if there were no node deletions. It ensures this property even while keeping the degree increase of all nodes no more than a multiplicative factor of 3. Moreover, we show that this tradeoff between stretch and degree increase is asymptotically optimal. Second, the Forgiving Graph handles both adversarial insertions and deletions, while the Forgiving Tree could only handle adversarial deletions (and no type of insertion). Finally, the Forgiving Graph does not require

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an initialization phase, while the Forgiving Tree required an initialization phase which involved sending $O(n \log n)$ messages, where n was the number of nodes initially in the network, and had a latency equal to the initial diameter of the network. Additionally, the Forgiving Graph is divergent technically from the Forgiving Tree, it makes significant use of a novel distributed data structure that we call a Half-full Tree (HaFT).

Our Model: We now describe our model of attack and network response, which is identical to that of [7]. We assume that the network is initially a connected graph over n nodes. An adversary repeatedly attacks the network. This adversary knows the network topology and our algorithm, and it has the ability to delete arbitrary nodes from the network or insert a new node in the system which it can connect to any subset of the nodes currently in the system. However, we assume the adversary is constrained in that in any time step it can only delete or insert a single node.

Our Results: For a peer-to-peer network that has both insertions and deletions, let G' be the graph consisting of the original nodes and inserted nodes without any changes due to deletions. Let n be the number of nodes in G'. The Forgiving Graph ensures that: 1) the distance between any two nodes of the actual network never increases by more than $\log n$ times their distance in G'; and 2) the degree of any node never increases by more than 3 times its degree in G'. Our algorithm is completely distributed and resource efficient. Specifically, after deletion, repair takes $O(\log d \log n)$ time and requires sending $O(d \log n)$ messages, each of size $O(\log n)$ where d is the degree of the node that was deleted. The formal statement and proof of these results is in Section 5.1.

Related Work: Our work significantly builds on work in [7] as described above. There have been numerous other papers that discuss strategies for adding additional capacity or rerouting in anticipation of failures [3, 4, 9, 13, 15, 16]. Results that are responsive in some sense include the following. Médard, Finn, Barry, and Gallager [10] propose constructing redundant trees to make backup routes possible when an edge or node is deleted. Anderson, Balakrishnan, Kaashoek, and Morris [1] modify some existing nodes to be RON (Resilient Overlay Network) nodes to detect failures and reroute accordingly. Some networks have enough redundancy built in so that separate parts of the network can function on their own in case of an attack [5]. In all these past results, the network topology is fixed. In contrast, our approach adds edges to the network as node failures occur. Further, our approach does not dictate routing paths or specifically require redundant components to be placed in the network initially. Our model of attack and repair builds on earlier work in [2, 14].

There has also been recent research in the physics community on preventing cascading failures. In the model used for these results, each vertex in the network starts with a fixed capacity. When a vertex is deleted, some of its "load" (typically defined as the number of shortest paths that go through the vertex) is diverted to the remaining vertices. The remaining vertices, in turn, can fail if the extra load exceeds their capacities. Motter, Lai, Holme, and Kim have shown empirically that even a single node deletion can cause a constant fraction of the nodes to fail in a power-law network due to cascading failures[8, 12]. Motter and Lai propose a strategy for addressing this problem by intentional removal of certain nodes in the network after a failure begins [11]. Hayashi and Miyazaki propose another strategy, called emergent rewirings, that adds edges to the network after a failure begins to prevent the failure from cascading[6]. Both of these approaches are shown to work well empirically on many networks. However, unfortunately, they perform very poorly under adversarial attack.

2 Node Insert, Delete and Network Repair Model

We now describe the details of our node insert, delete and network repair model. Let $G = G_0$ be an arbitrary graph on n nodes, which represent processors in a distributed network. In each step, the adversary either deletes or adds a node. After each deletion, the algorithm gets to add some new edges to the graph, as well as deleting old ones. At each insertion, the processors follow a protocol to update

Figure 1: The Node Insert, Delete and Network Repair Model – Distributed View.

Each node of G_0 is a processor.
Each processor starts with a list of its neighbors in G_0.
Pre-processing: Processors may send messages to and from their neighbors.
for $t := 1$ to T **do**
  Adversary deletes or inserts a node v_t from/into G_{t-1}, forming H_t.
  if node v_t is inserted **then**
    The new neighbors of v_t may update their information and send messages to and from their neighbors.
  end if
  if node v_t is deleted **then**
    All neighbors of v_t are informed of the deletion.
    Recovery phase:
    Nodes of H_t may communicate (asynchronously, in parallel) with their immediate neighbors. These messages are never lost or corrupted, and may contain the names of other vertices.
    During this phase, each node may insert edges joining it to any other nodes as desired. Nodes may also drop edges from previous rounds if no longer required.
  end if
  At the end of this phase, we call the graph G_t.
end for

Success metrics: Minimize the following "complexity" measures:
Consider the graph G' which is the graph consisting solely of the original nodes and insertions without regard to deletions and healings. Graph G'_t is G' at timestep t (i.e. after the t^{th} insertion or deletion).

1. **Degree increase.** $\max_{v \in G} \text{degree}(v, G_T)/\text{degree}(v, G'_T)$
2. **Network stretch.** $\max_{x,y \in G_T} \frac{dist(x,y,G_T)}{dist(x,y,G'_T)}$, where, for a graph G and nodes x and y in G, $dist(x, y, G)$ is the length of the shortest path between x and y in G.
3. **Communication per node.** The maximum number of bits sent by a single node in a single recovery round.
4. **Recovery time.** The maximum total time for a recovery round, assuming it takes a message no more than 1 time unit to traverse any edge and we have unlimited local computational power at each node.

their information. The algorithm's goal is to maintain connectivity in the network, keeping the distance between the nodes small. At the same time, the algorithm wants to minimize the resources spent on this task, including keeping node degree small.

Initially, each processor only knows its neighbors in G_0, and is unaware of the structure of the rest of G_0. After each deletion or insertion, only the neighbors of the deleted or inserted vertex are informed that the deletion or insertion has occured. After this, processors are allowed to communicate by sending a limited number of messages to their direct neighbors. We assume that these messages are always sent and received successfully. The processors may also request new edges be added to the graph. The only synchronicity assumption we make is that no other vertex is deleted or inserted until the end of this round of computation and communication has concluded. To make this assumption more reasonable, the per-node communication cost should be very small in n (e.g. at most logarithmic).

We also allow a certain amount of pre-processing to be done before the first attack occurs. This may,

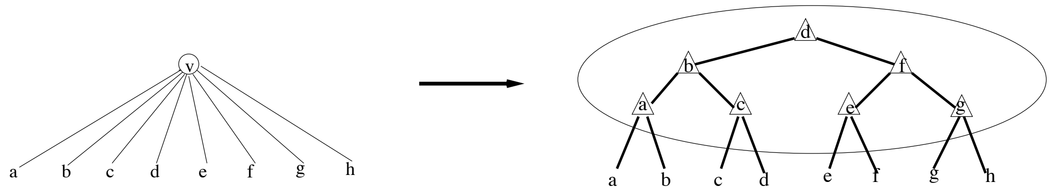


Figure 2: Deleted node v replaced by its Reconstruction Tree. The nodes in the triangle are helper nodes simulated by the real nodes which are in the leaf layer.

for instance, be used by the processors to gather some topological information about G_0, or perhaps to coordinate a strategy. Another success metric is the amount of computation and communication needed during this preprocessing round. Our full model is described in Figure 1.

3 The Forgiving Graph algorithm

At a high level, our algorithm works as follows:

In our model, an adversary can effect the network in one of two ways: inserting a new node in the network or deleting an existing node from the network. Node insertion is straightforward and is dependent on the specific policies of the network. When an insertion happens, our incoming node and its neighbors update the data structures that are used by our algorithm. We will also assume that nodes maintain neighbor-of-neighbor information.

Each time a node v is deleted, we can think of it as being replaced by a Reconstruction Tree (RT(v), for short) which is a haft (discussed in Section 4) having "virtual" nodes as internal nodes and neighbors of v as the leaf nodes. Note that each virtual node has a degree of at most 3. A single real node itself is a trivial RT with one node. RT(v) is formed by merging all the neighboring RTs of v using the strip and merge operations from Section 4. After a long sequence of such insertions and deletions, we are left with a graph which is a patchwork mix of virtual nodes and original nodes.

Also, because the virtual trees (hafts) are balanced binary trees, the deletion of a node v can, at worst, cause the distances between its neighbors to increase from 2 to $2\lceil \log d \rceil$ by travelling through its RT, where d is the degree of v in G' (the graph consisting solely of the original nodes and insertions without regard to deletions and healings). However, since this deletion may cause many RTs to merge and the new RT formed may involve all the nodes in the graph, the distances between any pair of actual surviving nodes may increase by no more than a $\lceil \log n \rceil$ factor.

Since our algorithm is only allowed to add edges and not nodes, we cannot really add these virtual nodes to the network. We get around this by assigning each virtual node to an actual node, and adding new edges between actual nodes in order to allow "simulation" of each virtual node. More precisely, our actual graph is the homomorphic image of the graph described above, under a graph homomorphism which fixes the actual nodes in the graph and maps each virtual node to a distinct actual node which is "simulating" it.

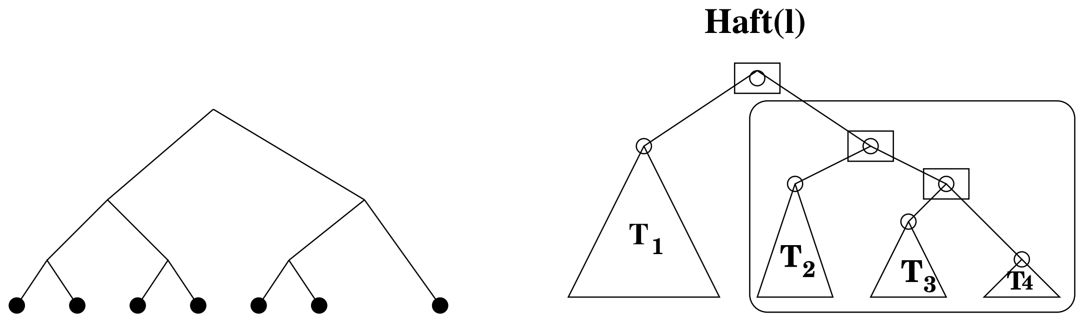
Note that, because each actual node simulates at most one virtual node for each of its deleted neighbors, and virtual nodes have degree at most 3, this ensures that the maximum degree increase of our algorithm is at most 3 times the node's degree in G'.

4 Half-full Trees

This section defines half-full trees (*haft*, for short), and describes some of their interesting properties of concern to us.

Half-full tree: A haft is a rooted binary tree in which every non-leaf node v has the following properties:

- v has exactly two children.



(a) A haft with 7 leaf nodes.

(b) A haft of n leaves. Every haft is a union of complete binary trees. In our notation, T_a is a complete binary tree and $|T_a|$ is the number of leaf nodes in T_a. The nodes in the square boxes are the nodes not part of a complete tree.

Figure 3: haft (half-full tree)

- The left child of v is the root of a complete binary subtree, containing half or more of v's descendants.

An example of a haft is shown in figure 3(a). For any positive l, there is a single unique haft over l leaf nodes (see lemma 1), that we refer to as as haft(l)

Lemma 1. *Let l be a positive integer. Then, the following are true:*

1. *There is a single unique* haft *with l leaf nodes, that we refer to as* haft(l)*.*
2. *binary representation (one-to-one correspondence): Let $a_k a_{k-1} ... a_0$ be the binary representation of n. Let h be the number of ones in this representation. Let $x_1, x_2, \dots, x_h$ be the indices of the one bits, and $n = \sum_{i=1}^{h} 2^{x_i}$, sorted in descending order. Let T_i be the complete binary tree with 2^{x_i} leaves. We can break* haft(l) *into a forest of h complete binary trees ($T_1, T_2, \dots T_h$) by removing $h-1$ nodes from T.*
3. *The depth of* haft(l) *is $\lceil \log l \rceil$.*

Proof. We now prove parts 1 and 2. Let T be a haft on l leaves. As a running example, consider the haft shown in Figure 3(b). Let $a_k a_{k-1} ... a_0$ be the binary representation of l. Let h be the number of ones in this representation. Let $x_1, x_2, \dots, x_h$ be the indices of the one bits sorted in descending order, and $l = \sum_{i=1}^{h} 2^{x_i}$. Let T_i be the complete binary tree with 2^{x_i} leaves. By definition of a haft, there are two cases:

1. *T is a complete tree*: This happens when $h = 1$ and $n = 2^{x_1}$. Clearly, T is unique, corresponding to the complete tree T_1.
2. *T is not a complete tree*: By definition of haft, the left child of the root is a complete tree and moreover this tree has half or more of the children of the root. Let $Size(X)$ be the number of nodes in a tree X. Since $Size(T_i) = 2^{x_i+1} - 1$ we know that $Size(T_1) > \sum_{k=2}^{h} Size(T_k)$. Thus, the complete tree to the left of the root has to be T_1.
   Applying the same definition to the right child of the root, we see that either this node heads the tree T_2, or its left subtree is T_2. Recursively applying this reasoning, we see that haft(l) is a unique tree with the trees T_1 to T_h joined by $h-1$ single nodes (For example in in Figure 3(b), these $h-1$ single nodes are marked as square boxes). It directly follows that removing these $h-1$ nodes leaves us with a forest of h complete binary trees $T_1, T_2, \dots T_h$.

For part 3, there are two possibilities:

1. *T is a complete tree:* For a complete tree with l leaves, we know that the depth of the tree is $\log l$.

2. *T is not a complete tree:* We show this by induction on the number of leaf nodes. Consider a haft with l leaf nodes. If $l = 1$, the haft is a complete tree so the height is 0, which is $\log l$. For larger l, we note that the left child of the root heads a complete subtree with less than l leaf nodes. Thus, the height of this left subtree is no more than $\log l$. Moreover, the right child of the root heads a haft over no more than $\frac{l}{2}$ leaf nodes. Thus, by the inductive hypothesis, this right subtree has height at most $\lceil \log \frac{l}{2} \rceil$. Thus, the height of haft(l) is $1 + max(\log x, \log(l - x))$, where x is a power of 2 and $\frac{l}{2} \leq x < n$. Since $x > l - x$, it follows that $\log x = \lceil logx \rceil \geq \lceil \log(n - x) \rceil$. Finally, the height of haft(l) is $1 + \log x = \lceil \log l \rceil$, since $\frac{l}{2} \geq x < l$.

□

4.1 Operations on Hafts

We Define the following operations on hafts:

1. *Strip*: Suppose T is a haft with h ones in its binary representation. The Strip operation removes $h - 1$ nodes from T returning a forest of h complete trees.

2. *Merge*: The Merge operation joins hafts together using additional isolated single nodes, to create a single new haft.

We now describe these operations in more detail:

4.1.1 Strip

The operation Strip(T) takes a haft T and returns a forest F, of complete trees. As follows from part 2 of lemma 1, each haft can be broken into a forest of h complete trees where h is the number of ones in the binary representation of the number of leaves of T. We call the roots of these complete trees primary roots. Before we proceed further, let us formally define this concept:

Primary root: A primary root is a node in a haft that has the following properties:

- It is the root of a complete subtree.
- Its parent, if it has one, is not the root of a complete subtree.

The Strip operation works as follows: If T is a complete tree, then return T itself. Note that the root of the T is the only primary root in this case. If T is not a complete tree, then F is obtained as follows: Starting from the root of T, traverse the direct path towards the rightmost leaf of T. Remove a node if it is not a primary root. Stop when a primary root or a leaf node (which is a primary root too) is discovered. In figure 3(b) the Strip operation removes the nodes indicated by the square boxes.
We now give intuition as to why the Strip operation works.

Lemma 2. *The Strip operation returns the subtrees rooted at all primary roots in the input* haft*.*

Proof. By the definitions of haft and primary root, if a vertex is not the root of a complete subtree, its left child is guaranteed to be a primary root. Thus, either the root of the haft is a primary root or its left child is. If the left child is a primary root, there can be no other primary root in the left subtree, so we we return the tree rooted at that child. Recursively applying the same test to the right child, we get all the primary roots. □

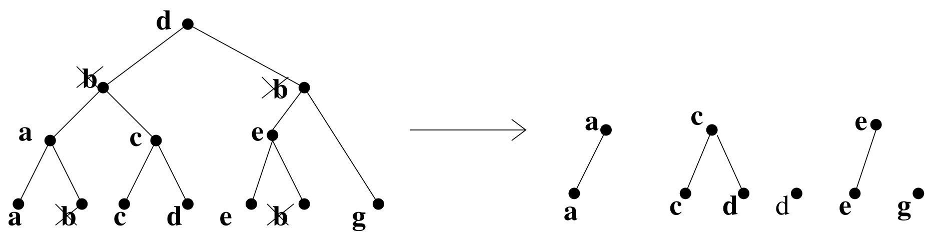


Figure 4: Deletion of a node and its helper nodes lead to breakup of RT into components. The Strip operation or a simple variant (for non-hafts) returns a set of complete trees, which can then be merged.

4.1.2 Merge

Every haft can be represented as a binary number (by lemma 1). Merging hafts is analogous to binary addition of the binary number representations of these trees. The new binary number obtained is the representation of the half-full tree corresponding to the merge. This is illustrated in figure 5.

The first step of the Merge operation is to apply the Strip operation on the input trees. This gives a forest of complete trees. These complete trees can be recombined with the help of extra nodes to obtain a new haft. Let $Size(X)$ be the number of nodes in a tree X. Consider two complete trees T_1 and T_2 (Size(T_1) $> Size(T_2)$), with roots r_1 and r_2 respectively, and an extra node v. To merge these trees, make r_1 the left child and r_2 the right child of v by adding edges between them. The merged tree is always a haft. Thus, the merge operation Merge($\text{haft}_1, \text{haft}_2, \ldots$) is as follows:

1. Apply Strip to all the hafts to get a forest of complete trees.
2. Let $T_1, T_2, \ldots, T_k$ be the k complete trees sorted in ascending order of their size. Traverse the list from the left, let T_i and T_{i+1} be the first two adjacent trees of the same size and v be a single isolated vertex, join T_i and T_{i+1} by making v the parent of the root of T_i and the root of T_{i+1}, to give a new tree. Reinsert this tree in the correct place in the sorted list. Continue traversal of the list from the position of the last merge, joining pairs of trees of equal sizes. At the end of this traversal, we are left with a sorted list of complete trees, all of different sizes.
3. Let $T_1, T_2, \ldots, T_l$ be the sorted list of complete trees obtained after the previous step. Traverse the list from left to right, joining adjacent trees using single isolated vertices. Let w be a single isolated vertex. Join T_1 and T_2 by making the root of T_2 the left child and the root of T_1 the right child of w, respectively. This gives a new haft. Join this haft and T_3 by using another available isolated vertex, making the larger tree (T_3) its left child. Continue this process till there is a single haft.

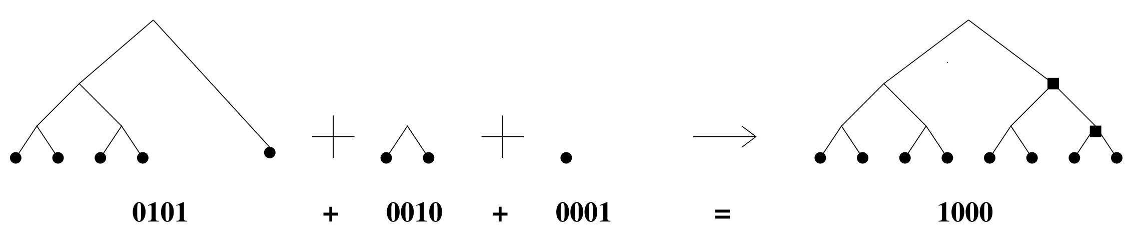


Figure 5: Merging three hafts. The square shaped vertices are the isolated vertices used to join complete trees. Merging is analagous to binary number addition, where the number of leaves are represented as binary numbers.

4.2 Detailed description

As mentioned earlier, deletion of a node v leads to it being replaced by a Reconstruction Tree (RT(v), for short) in G (Refer to Table 1 for definitions). The RT is a haft (discussed in Section 4) having "virtual" nodes as internal nodes and neighbors of v as the leaf nodes. The real network is a homomorphic image of this virtual graph. The nodes in the virtual graph refer to the corresponding processor in the network, as shown in Figure 6. The nodes in G corresponding to an edge of v in G' and forming the leaf nodes in any RT are called real nodes, and those internal to a RT and simulated by the real nodes (more precisely, by the processor) are called helper nodes. There is one real node and at most one helper node corresponding to an edge of v in G' i.e. to an edge formed when v or v's neighbor joined the network. In Table 1 we list the information each processor v requires for each edge in order to execute the ForgivingGraph algorithm. When one of the nodes of the edge gets deleted, in G, that node may be replaced by a helper node. This end point of the edge is stored in the field v.endpoint. For an edge (v, x), if x is a real node then the field v.endpoint is simply the node x. If the node x gets deleted, the new endpoint may be a helper node, though we still refer to this edge as (v, x) i.e. by its name in G'. Moreover, the processor may now simulate a helper node corresponding to this edge. Since each edge is uniquely identified, the real nodes and helper nodes corresponding to that edge can also be uniquely identified. This identification is used by the processors to pass messages along the correct paths. The Forgiving graph algorithm is given in pseudocode form in Algorithm A.1 alongwith the required subroutines. For ease of description, the real and helper nodes belonging to the same processor may not be explicitly distinguished in the code.

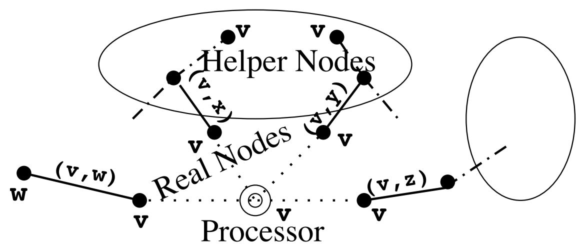


Figure 6: The Nodes corresponding to the processor v in the graph G. An ellipse denotes a RT created on deletion of a neighbor of v.

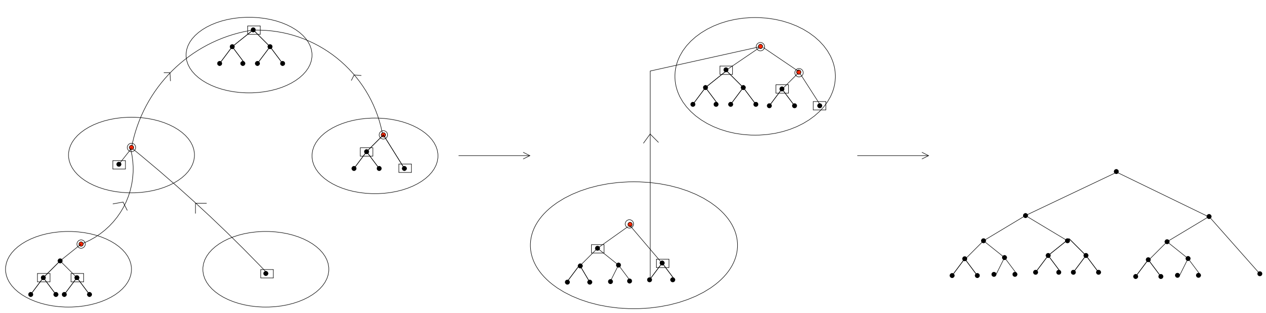


Figure 7: On deletion of a node v, The RTs to be merged are connected by BT_v which is a binary tree. The RTs merge from the bottom up with their parents till a single RT is left. The nodes in the square boxes are the primary roots. The (red color) nodes in the circle are excess nodes removed at each step.

On deletion of a node, the repair proceeds in two phases. The first phase is a quick $O(1)$ phase in which the neighbors of the deleted node connect themselves in the form of a binary tree (Algorithm A.3). These neighbors represent the independent components created on deletion of the node. Some of these components may not be hafts. We shall refer to such a subtrees as a RTfragment. Let v be the processor deleted. Then, we call this tree formed by the v's neighbors as BT_v and the nodes forming BT_v as

Processor v: Edge(v,x)	
Real node fields	
`Endpoint`	The node that represents the other end of the edge. For edge(v,x) this will be node x if x is alive or RTparent if x is not.
`hashelper`	(boolean field). True if there is a helper node simulated by v corresponding to this edge.
`RTparent`	Parent of v in RT. Non NULL only if x has been deleted.
`Representative`	This is v itself. Field used during merging of RTs.
Helper node fields	Fields for helper node corresponding to the edge. Non NULL only if the helper node exists. Sometimes, we will refer to a helper field as *edge.helper.field*
`hparent`	Parent of helper node.
`hrightchild`	Right Child of helper node.
`hleftchild`	Left Child of helper node.
`height`	Height of the helper node.
`childrencount`	The number of descendants of the helper node.
`Representative`	The unique leaf node of a subtree of a RT that does not have a helper node in that subtree. This node is used during merging of RTs.

Table 1: The fields maintained by a processor v for edge(v, x), which is an edge in G', the graph of only original nodes and insertions.

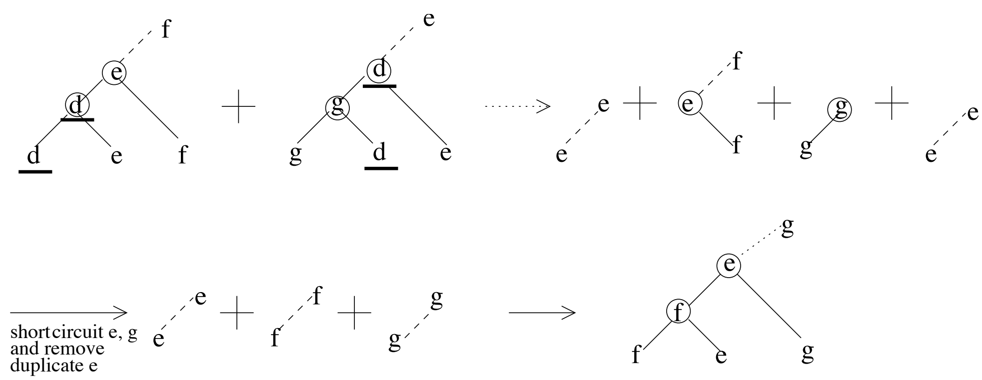


Figure 8: The underlined node d and corresponding helpers are deleted. This leads to the graph breaking into components which are then merged using BT_d (the binary tree of anchors) and the primary roots in the components. The dashed edges show the representative for that node.

anchors. Formally, we define an anchor as follows:

Anchor : An anchor is the unique designated node in a RT or RTfragment that takes part in the binary tree BT_v.

In phase 2, the RTs and RTfragments forming BT_v have to be merged (Figure 7). We are only interested in the complete trees in these since we can discard all other helper nodes. The anchors send probe messages to discover the primary roots which head these complete trees (Algorithm 4). This is similar to the Strip operation described in Section 4.1.1. The nodes maintain information about their height and number of their children in their RT or RTfragment. Thus, they are able to identify themselves as primary roots. At the same time, the nodes outside the complete trees are identified and marked for removal. The complete trees are then merged pairwise in a bottomup fashion till only a single haft remains. This is illustrated in figure 7. At each round, every leaf RT in BT_v will merge with its parent

RT. This can be done in parallel, so that the number of rounds of merges will be equivalent to the height of the tree. For two trees to merge, as shown in the Merge operation (Section 4.1.2), an additional node is needed that will become the parent of these two trees. This node must be simulated by a real node that is not already simulating a helper node in the trees. Since the number of internal nodes in a tree is one less than the leaf nodes, there is exactly one such leaf node for each tree. The roots of these two trees keep the identity of this node. This is stored in the field Representative (Table 1). More formally, we define a representative as follows:

Representative Given a node y, the representative of y is a real node, decided as follows:

- If y is a real node, then y itself.
- If y is a helper node, then the unique leaf node of y's subtree in y's RT that does not have a helper node in that subtree.

We now describe a mechanism for merging that we call the representative mechanism. Each node has a representative defined earlier. When two trees (Note that a tree may even be a single node) are merged (Algorithm A.8 and Algorithm A.9), the representative of the root of the bigger tree (or of one of the trees, if they have the same size) instantiates a new helper node, and makes the two roots its children. The new helper node will now inherit as its representative the representative of the root of the other tree, since this is the node in the merged tree that does not have a helper node in the tree. An example of merging using this mechanism is shown in Figure 8. At the end of each round, we have a new set of leaf RTs. Each new leaf is now a merged haft of the previous leaves and their parent. We need a new anchor for this haft. We can continue having the anchor of the parent RT or RTfragment as the anchor. However, this node may be one of the extra nodes marked for removal. In this case, the anchor designates one of the nodes that was a primary root in its RT as the new anchor, passes on its links and removes itself. Now, the newly formed leaf hafts may have primary roots which are different from those of the previous ones. The new anchor will again send probe messages and gather this information and inform the new primary roots of their role. This process will continue till we are left with a single RT. This is shown in Figure 7.

5 Results

5.1 Upper Bounds

Let G' be the graph consisting solely of the original nodes and insertions without regard to deletions and healings. Let G_T and G'_T be the graphs at time T.

Lemma 3. *Given the real node v in G corresponding to an edge (v, x) in G',*

1. *There can be at most one helper node in G corresponding to v.*
2. *During the Repair phase, there can be at most two helper nodes corresponding to the edge (v, x). Moreover, one of these could also be an anchor in BT_v*

Proof. As stated earlier, there is only one real node in G corresponding to an edge in G' (Figure 6). Also, any real node can only form a leaf node of a RT, and a helper node can only be an internal node. We prove part 1 by contradiction. Suppose there are two helper nodes in G corresponding to the real node v. Let us call these nodes v' and v''. The following cases arise:

1. *v' and v'' belong to different* RT*s:*
   By the representative mechanism, a helper node is created only if the real node that simulates it is the representative of a node (e.g. in line 7 in Algorithm A.9). By definition, the representative of a node is a unique leaf node in the subtree headed by that node in its RT. If both v' and v'' exist and belong to different RTs, this implies that node v exists as a leaf node in two different RTs. This is a contradiction.

2. *v' and v'' belong to the same* RT*:*
   Without loss of generality, assume that the $v''.height \geq v'.height$. The following cases arise:
   (a) *v' is a node in the subtree headed by v'':* Note that by the representative mechanism, in a subtree, an internal helper node will be created earlier than the root of the subtree. Thus, node v' will be created before node v''. Let node y be the child of node v'' that had y.Representative $=$ v'' when v'' was created. However, y.Representative could not have been v'', since by definition, y.Representative has to be the unique leaf node not simulating a helper node in y's subtree, but v is already simulating v' in v'''s subtree.
   (b) *v' is a node not in the subtree headed by v'':* Again, the representaive mechanism and definition of a representative implies that node v was a representative in two non-intersecting subtrees in the same RT. This implies that node v occurs as a leaf twice in that RT. This is not possible.

Now, we prove part 2. As stated earlier, at each stage of the merge procedure, leaf RTs or RTfragments in BT_v will merge with their parent. Suppose that v' is a helper node simulated by real node v, and v' is not part of any complete subtree in such a RTfragment or RT. This means that v' will be marked red and removed when this stage of merge is completed (Refer Figure 7). Let node y be the root of the complete subtree (i.e. a primary root in that RTfragment) that has v as a leaf node. Node v' is an ancestor of node y since v' cannot a descendant. By definition, y.Representative $= v$, since v will be the unique leaf node in y's subtree not simulating a helper node in that subtree. When the trees are being merged, v may be asked to create another helper node. Thus, v may have two helper nodes. Also, each RT or RTfragment has exactly one anchor node. This anchor may be v' or another node. Thus, in the repair phase, a real node may simulate upto two helper nodes, and one of these helper nodes may be an anchor. However, node v' will be removed as soon as this stage is completed, and if v' was an anchor, a new anchor is chosen from the existing nodes. Since at the end of the merge, BT_v collapses to leave one RT, the extra helper nodes and the edges from the anchor nodes are not present in G, thus, not contradicting part 1.

□

Lemma 4. *After each deletion, the repair can take $O(\log d \log n)$ time to exchange $O(d \log n)$ messages of size $O(logn)$, where d is the degree of the deleted node.*

Proof. There are mainly two types of messages exchanged by the algorithm. They are the probe messages sent by the FindPrRoots() (Algorithm A.5) within a RT and the messages containing the information about the primary roots exchanged by the anchors in BT_v and among the primary roots themselves (Algorithm A.7: ComputeHaft()). Let size(BT_v) be the number of RTs of BT_v. Since a helper node can split a RT into maximum 3 parts, and there can be at most d helper nodes, where d is the degree of the deleted node v, size(BT_v) $= 3d$. Now, let us calculate the number of messages:

- *Probe messages (Algorithm A.5)*: A probe message is generated by a an anchor of a RT. This is similiar to the *Strip* operation (Section 4.1.1). The path that the probe message follows is the direct path from the originating node to the rightmost node of the RT. At the most 2 messages can be generated for every node on the way. Further, there can be one confirmatory message transmitted from the primary roots back to the anchor. Let *numnodes* be the number of nodes and *numprobes* be number of probe messages sent in a single RT. Thus,

$$\begin{aligned} numprobes &\leq 3 \log numnodes \\ &\leq 3 \log n \end{aligned}$$

- *Exchange of primary roots lists (Algorithm A.7)*: At each step of Algorithm A.4 (BottomupRTMerge()), leaves in BT_v merge with their parents. Let *rtlistmsgs* be the number of messages exchanged for every such merge. The anchors of the leaves of BT_v send their primary roots lists to the parent, which in turn can send both it's list and the sibling's list to the child. Thus, $rtlistmsgs = 4$. In addition, every anchor will send this list to the primary roots in its RT, generating at most another $\log n$ messages (Let us call this $AtoRmsgs$).

As stated earlier, in the BT_v, leaves merge with their parents. The number of such merges before we are left with a single RT is $\lceil \text{size}(BT_v)/2 - 1 \rceil$. Also, at most 3 RTs are involved in each merge. Let $totmessages$ be the total number of messages exchanged. Hence,

$$\begin{aligned} totmessages &= \lceil \text{size}(BT_v)/2 - 1 \rceil (3(numprobes + AtoRmsgs) + rtlistmsgs) \\ &\leq \lceil 3d/2 - 1 \rceil (12 \log n + 4) \\ &\in O(d \log n) \end{aligned}$$

In BT_v, leaves and their parents merge. This can be done in parallel such that each time the level of BT_v reduces by one. Within each RT, the time taken for message passing is still bounded by $O(\log n)$ assuming constant time to pass a message along an edge. Since there are at most $\lceil logd \rceil$ levels, the time taken for passing the messages is $O(\log d \log n)$. The biggest message exchanged may have constant size information about the primary roots of upto two RTs. This may be the message sent by a parent RT in BT_v to its children RT. Since there can be at most $O(\log n)$ primary roots, the size of messages is $O(\log n)$. □

Here, we state our main theorem.

Theorem 1. *The Forgiving Graph has the following properties:*

1. Degree increase: *For any node v, $d(v, G_T) \leq 3 \times d(v, G'_T)$, where d is the degree of the node v.*

2. Stretch: *For any pair of nodes x and y, $distance(x, y, G_T) \leq (\log n) \times distance(x, y, G'_T)$.*

3. Cost: *After each deletion, the repair can take upto $O(\log d \log n)$ time with $O(d \log n)$ messages of size upto $O(logn)$, where d is the degree of the deleted node.*

Proof. Part 1 follow directly by construction of our algorithm. For part 1, we note that for a node v, any degree increase for v is imposed by the edges of its helper node to hparent(v) and hchildren(v). From lemma 3 part 1, we know that, in G, node v can play the role of at most one helper node for any of its neighbors in G' at any time (i.e. $d(v, G'_T)$). The number of hchildren of a helper node are never more than 2, because the reconstruction trees are binary trees. Thus the total degree of v ($d(v, G_T)$) is at most 3 times its degree in G' ($d(v, G'_T)$).

We next show Part 2, that the stretch of the Forgiving Graph is $O(D \log n)$, where n is the number of nodes in G_T. The distance between any two nodes x and y cannot increase by more than the factor of the longest path in the largest RT on the path between x and y. This factor is $\log n$ at the maximum.

Part 3 follows from Lemma 4. Note that besides the commuication of the messsages discussed, the other operations can be done in constant time in our algorithm.

□

5.2 Lower Bounds

Theorem 2. *Consider any self-healing algorithm that ensures that: 1) each node increases its degree by a multiplicative factor of at most* α*, where* $\alpha \geq 3$*; and 2) the stretch of the graph increases by a multiplicative factor of at most* β*. Then, for some initial graph with* n *nodes, it must be the case that* $\beta \geq \frac{1}{2}\log_{\alpha-1}(n-1)$*.*

Proof. Let G be a star on n vertices, where x is the root node, and x has an edge with each of the other nodes in the graph. The other nodes (besides x) have a degree of only 1. Let G' be the graph created after the adversary deletes the node x. Consider a breadth first search tree, T, rooted at some arbitrary node y in G'. We know that the self-healing algorithm can increase the degree of each node by at most a factor of α, thus every node in T besides y can have at most $\alpha - 1$ children. Let h be the height of T. Then we know that $1 + \alpha \sum_{i=0}^{h-1} (\alpha-1)^i \geq n-1$. This implies that $(\alpha-1)^h \geq n-1$ for $\alpha \geq 3$, or $h \geq \log_{\alpha-1}(n-1)$. Let z be a leaf node in T of largest depth. Then, the distance between y and z in G' is h and the distance between y and z in G is 2. Thus, $\beta \geq h/2$, and $2\beta \geq \log_{\alpha-1}(n-1)$, or $\beta \geq \frac{1}{2}\log_{\alpha-1}(n-1)$. □

We note that this lower-bound compares favorably with the general result achieved with our data structure.

6 Conclusion

We have presented a distributed data structure that withstands repeated adversarial node deletions by adding a small number of new edges after each deletion. Our data structure is efficient and ensures two key properties, even in the face of both adversarial deletions and adversarial insertions. First, the distance between any pair of nodes never increases by more than a $\log n$ multiplicative factor than what the distance would be without the adversarial deletions. Second, the degree of any node never increases by more than a 3 multiplicative factor.

Several open problems remain including the following. Can we design algorithms for less flexible networks such as sensor networks? For example, what if the only edges we can add are those that span a small distance in the original network? Can we extend the concept of self-healing to other objects besides graphs? For example, can we design algorithms to rewire a circuit so that it maintains its functionality even when multiple gates fail?

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A ForgivingGraph PseudoCode

```
Given a Graph G(V, E)
Require: each node of G has a unique ID
for each node v ∈ G do
   Init(v).
end for
while true do
   if a vertex v is inserted then
      vertex v and new neighbors add appropriate edges.
      Init(v).
   else if a vertex v is deleted then
      DeleteFix(v)
   end if
end while
```

Algorithm A.1: Forgiving graph: The main function.

```
for each edge(v, x) do
   (v, x).Representative = v
   set other fields to NULL.
end for
```

Algorithm A.2: Init(v): initialization of the node v

```
Nset = {}
for each edge(v, x) do
   if (v, x).hashelper = TRUE then
      Nset = Nset ∪ (v, x).hparent ∪ (v, x).hrightchild
   end if
   Nset = Nset ∪ (v, x).endpoint
end for
The nodes in Nset make new edges to make a balanced binary tree BT_v(Nset, E_v).
BottomupRTMerge(BT_v, v)
delete the edges E_v.
```

Algorithm A.3: DeleteFix(v): Self-healing on deletion of a node

```
if BT_v has only one node then
  return
end if
for y ∈ BT_v do
  if y is a real node then
    Let PrRoots(y) ← y
  else if y = (v, x).endpoint then
    FindPrRoots(y, 1, real(v), TRUE )
  else if helper(y).hparent = v OR helper(y).hleftchild = v OR helper(y).hrightchild = v then
    Let PrRoots(y) ← FindPrRoots(y, v.childrencount, helper(v), TRUE)
  else
    Let PrRoots(y) ← FindPrRoots(y, v.childrencount, helper(v), FALSE)
  end if
end for
for all nodes y s.t. node y is a parent of a leaf in BT_v do
  if y has two children in BT_v then
    Haft_Merge(y, y 's left child in BT_v, y 's right child in BT_v)
  else
    Haft_Merge(y, y's left child, NULL)
  end if
end for
BottomupRTMerge(BT_v) // The new leaf nodes merge again till only one is left.
```

Algorithm A.4: BottomupRTMerge(BT_v, v): The nodes of BT_v merge their RTs starting from the leaves going up forming a new BT_v.

```
if Breakflag = TRUE AND (sender = y.hrightchild OR sender = y.hleftchild ) then
  y.childrencount = y.childrencount - numchild
end if
if y.childrencount = 2^(y.height) then
  if TestPrimaryRoot(y) = TRUE then
    return {y,FindPrRoots(y.hparent, 0, y, Breakflag) }
  else
    return {FindPrRoots(y.hparent, 0, y, Breakflag) } // Node itself not a primary root but
    parent maybe.
  end if
else
  mark node red
  if exists(y.hleftchild) AND sender != y.hleftchild then
    FindPrRoots(y.hleftchild, y.childrencount, y, Breakflag)
  else if exists(y.hrightchild) AND sender != y.hrightchild then
    FindPrRoots(y.hrightchild, y.childrencount, y, Breakflag)
  else if exists(y.hparent) AND sender != y.hparent then
    FindPrRoots(y.hparent, y.childrencount, y, Breakflag)
  end if
end if
```

Algorithm A.5: FindPrRoots(y, numchild, sender, Breakflag): Find the primary roots in the RT beginning with node y. If Breakflag is set the tree is a component of the RT formed due to the deletion.

if y.childrencount $= 2^{y.\text{height}}$ **then**
    if y.hparent $= NULL$ **then**
        return TRUE
    else if y.hparent.childrencount $\neq 2^{y.\text{hparent.height}}$ **then**
        return TRUE
    end if
end if
return FALSE

Algorithm A.6: TestPrimaryRoot(y): Tell if helper node y is a primary root in RT

Nodes p, l and r exchange PrRoots(p), PrRoots(l), PrRoots(r)
Let RT ← MakeRT(PrRoots(p), PrRoots(l), PrRoots(r))
if p is marked red **then**
    p transfers its edges in BT_v to one of PrRoots(p) // p needs to be removed, BT_v needs to be maintained
end if
Remove all helper nodes marked red // Some helper nodes marked red may have been reused and unmarked by MakeRT

Algorithm A.7: Haft_Merge(p, l, r): Merge the hafts mediated by anchors p, l and r

for all $y \in$ (PRoots1 $\cup$ PRoots2 $\cup$ PRoots3) **do**
    Let HaftMergePrint ← ComputeHaft(PRoots1, PRoots2, PRoots3)
    Make helper nodes and set fields and make edges according to HaftMergePrint
end for

Algorithm A.8: MakeRT(PRoots1, PRoots2, PRoots3): The sets of Primary roots make a new RT

Let $R = \text{PRoots1} \cup \text{PRoots2} \cup \text{PRoots3}$
Let $L = R$ sorted in ascending order of number of children, NodeID
Suppose L is $(r_1, r_2, \ldots, r_k)$ where the r_i are the k ordered primary roots.
set $ctr = 1, count = k$
while $ctr < count$ **do**
  if r_{ctr}.numchildren $= r_{ctr+1}$.numchildren **then**
    Make helper node helper(r_{ctr}.Representative). Initialise all its fields to NULL.
    Make helper(r_{ctr}.Representative) the parent of r_{ctr} and r_{ctr+1}
    if r_{ctr} is a real node **then**
      Set helper(r_{ctr}.Representative).height = 1
    else
      Set helper(r_{ctr}.Representative).height $= 2r_{ctr}.height$
    end if
    Set helper(r_{ctr}.Representative).Representative $= r_{ctr+1}$.Representative
    remove r_{ctr}, r_{ctr+1} and insert helperr_{ctr}Representative in the correct position in L.
    set $ctr \leftarrow ctr - 1$, $count \leftarrow count - 1$
  end if
  set $ctr \leftarrow ctr + 1$,
end while
set $ctr = 1$
while $ctr < count$ **do**
  Make helper node helper(r_{ctr+1}.Representative). Initialise all its fields to NULL
  Set helper(r_{ctr+1}.Representative).hleftchild $= r_{ctr+1}$
  Set helper(r_{ctr+1}.Representative).hrightchild $= r_{ctr}$
  Set helper(r_{ctr+1}.Representative).height $= r_{ctr+1}$.height + 1
  Set helper(r_{ctr+1}.Representative).Representative $= r_{ctr}$.Representative
  In L, replace r_{ctr+1} by helper(r_{ctr+1}.Representative)
end while

Algorithm A.9: ComputeHaft(PRoots1, PRoots2, PRoots3): (Implementation of Haft Merge) The primary roots compute the new haft