On the poset structure of *n*-potent right ideal commutative BCK-algebras

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BCK logic

- **BCK logic,** in symbols *BCK*, is the deductive system with language $\{\to\}$ presented by the axioms and inference rules

  $(p \to q) \to (q \to r) \to (p \to r)$ (B)

  $p \to (q \to r) \to q \to (p \to r)$ (C)

  $p \to (q \to p)$ (K)

  $p, p \to q \vdash_{BCK} q$ ($MP_{\to}$)

- BCK logic is regularly algebraisable witness $\{p \to q, q \to p\}$.
  - The class of all BCK-algebras is the equivalent quasivariety of *BCK*.

For any $\Gamma \cup \alpha \subseteq Fm_{\{\to\}}$, $\Gamma \vdash_{BCK} \alpha$ iff $\{\beta \approx \mathbf{1} : \beta \in \Gamma\} \vdash_{BCK} \alpha \approx \mathbf{1}$.

Order algebraisability

- Algebraisability is an abstract theory of the generalised biconditional $\leftrightarrow$ that associates logics with their 'natural' algebraic counterparts.
- **Question** (Pigozzi). Is there a theory of implication as abstract as the theory of algebraisability, but in which the role of equality is replaced by a suitable notion of order?
- *CPC* and *IPC* suggest 'yes', since

$$\varphi \vdash_{CPC} \psi \text{ iff } \vdash_{CPC} \varphi \to \psi \text{ iff } \models_{\mathbf{BA}} \varphi \to \psi = 1 \text{ iff } \varphi \leq \psi.$$

- BCK logic suggests 'no':

What plays the role of $\leq$ in this case?

Orders on BCK-algebras

- A **BCK-algebra** is an algebra $\langle A; *, 0\rangle$ satisfying the identities and quasi-identity:

$$((x * y) * (x * z)) * (z * y) \approx \mathbf{0}$$

$$x * \mathbf{0} \approx x$$

$$x * x \approx \mathbf{0}$$

$$x * y \approx \mathbf{0} \ \&\ y * x \approx \mathbf{0} \supset x \approx y.$$

  - (Hereafter we denote $*$ by juxtaposition. We also move schizophenically between algebraic and logical notation.)

- The class **BCK** of all BCK-algebras is a proper quasivariety.

- For any BCK-algebra **A**, the relation $\leq^0$ defined $\forall\ a, b \in A$ by $ab = 0$ is a partial order on A.

The Guzmán order and its generalisations

- **Theorem** (Guzmán). For any BCK-algebra **A**, the relation $\leq^1$ defined $\forall\ a, b \in A$ by $a \leq^1 b$ iff $a \in \{bc : c \in A\}$ is a partial order on A, which is finer than $\leq^0$.
- **Lemma.** For any BCK-algebra **A** and a, b in A, $a \leq^1 b$ iff $b(ba) = a$.
- Define the groupoid terms xy^n, $n \geq 1$, by $xy^0 = x$ and $xy^{k+1} = (xy^k)y$ for $k \geq 1$. Also, for any BCK-algebra **A**, let $bA^n = \{bc^n : c \in A\}$.
- Two obvious ways to generalise Guzmán's order are:
  - To the family $\{\subseteq^n : 1 \leq n \in \omega\}$, defined $\forall\ a, b \in A$ by

    $$a \subseteq^n b \text{ iff } b(ba)^n = a.$$

  - To the family $\{\leq^n : 1 \leq n \in \omega\}$, defined $\forall\ a, b \in A$ by

    $$a \leq^n b \text{ iff } a \in bA^n.$$

The family $\{\subseteq^n : 1 \leq n \in \omega\}$

- **Proposition.** For any BCK-algebra **A**,
  - (i) The relation $\subseteq^2$ defined $\forall\ a, b \in A$ by $a \subseteq^2 b$ iff $b(ba)^2 = a$ is a partial order on A.
  - (ii) If $a \subseteq^2 b$ then $a \subseteq^1 b$.
  - (iii) If $a \subseteq^2 b$ then $ac \subseteq^2 bc$.
  - (iv) $0 \subseteq^2 a$.
- **Proposition.** $\subseteq^1 = \leq^1$, while for each integer $n > 2$, $\subseteq^n = \subseteq^2$.

So we can put aside $\subseteq^n$, $n > 2$, from further consideration!

The family $\{\leq^n : 1 \leq n \in \omega\}$

- **Proposition.** For any BCK-algebra **A**,

  (i) For each integer $n \geq 0$, the relation $\leq^n$ defined $\forall\ a, b \in A$ by $a \leq^n b$ iff $a \in bA^n$ is a partial order on A.

  (ii) If $a \leq^n b$ then $a \leq^i b$, for $i = 0, 1$.

  (iii) If $a \leq^n b$ then $ac \leq^n bc$.

  (iv) $0 \leq^n a$.

  (v) If $a \subseteq^2 b$ then $a \leq^n b$ for any integer $n \geq 0$.

  (vi) $a \leq^1 b$ iff $a \subseteq^1 b$.

The order structure induced by the family $\{\leq^n : 0 \leq n \in \omega\}$ is generally quite complicated.

Order-theoretic characterisations of BCK-algebras

- A BCK-algebra is
  - **commutative** if $x(xy) \approx y(yx)$
  - **positive implicative** if $(xy)y \approx xy$
  - **implicative** if $x(yx) \approx x$
  - ***n*+1-potent** if $xy^{n+1} \approx xy^n$.

These are the "basic" varieties of BCK-algebras.

- **Theorem.** A variety **V** of BCK-algebras is
  (1) Commutative iff $\leq^0 = \leq^1$ for all $\mathbf{A} \in \mathbf{V}$
  (2) Positive implicative iff $\leq^1 = \subseteq^2$ for all $\mathbf{A} \in \mathbf{V}$
  (3) n+1-potent iff $\leq^n = \subseteq^2$ for all $\mathbf{A} \in \mathbf{V}$
  (4) Implicative iff $\leq^0 = \leq^1 = \subseteq^2$ for all $\mathbf{A} \in \mathbf{V}$.

The local deduction theorem in BCK logic

- *BCK* does not have the deduction-detachment theorem (DDT)
  - since the equivalent quasivariety of *BCK* does not have EDPC.
- *BCK* has the *local* deduction-detachment theorem (LDDT):

$\Gamma, \alpha \vdash \beta$ iff $\exists n \in \omega$ s.t. $\Gamma \vdash \alpha \rightarrow^n \beta$

iff $\exists n \in \omega$ s.t. $\Gamma \approx \mathbf{1} \models_{\mathbf{BCK}} \alpha \rightarrow^n \beta \approx \mathbf{1}$ (*)

  - The proof (by induction) does not explain *why BCK* has the LDDT.

But (*) encourages us to look at term reducts of BCK-algebras of the form $\langle A; \rightarrow^n, 1\rangle$.

BCS-algebras

- A **BCS-algebra** is an algebra $\langle A; -, 0\rangle$ satisfying the identities and quasi-identity:

$$x - \mathbf{0} \approx x$$
$$x - x \approx \mathbf{0}$$
$$(x - y) - z \approx (x - z) - y$$
$$(x - y) - z \approx (x - z) - (y - z).$$

  – We denote the class of all BCS-algebra by **BCS**.

- **Theorem.** A quasivariety of BCK-algebras satisfies (E_n) iff every algebra $\mathbf{A} \in \mathbf{V}$ has a BCS-algebra term reduct $\langle \mathrm{A}; -, 0\rangle$, where $a - b := ab^n$ for all $a, b \in A$.

1-assertional logics

- For a quasivariety **K** in a signature Λ with a constant term **1**, the **1**-assertional logic of **K**, in symbols $S(\mathbf{K}, \mathbf{1})$, is the consequence relation $\vdash_{S(\mathbf{K},\mathbf{1})} : \wp(\Lambda) \times \Lambda$ given by

$$\Gamma \vdash_{S(\mathbf{K},\mathbf{1})} \alpha \text{ iff } \Gamma \approx \mathbf{1} \models_{\mathbf{K}} \alpha \approx \mathbf{1}.$$

- **Proposition.** $S(\mathbf{BCS}, \mathbf{1}) = IPC^{\rightarrow}$.
- **Corollary.** Let **K** be a quasivariety of BCK-algebras. Then $S(\mathbf{K}, \mathbf{1})$ has the DDT iff $S(\mathbf{K}^{\rightarrow}, \mathbf{1})$ is $IPC^{\rightarrow}$.
  - Here $\mathbf{K}^{\rightarrow}$ denotes the class of all $\langle \rightarrow^{n}, 1\rangle$-term reducts of members of **K**.

Question: What is the connection with the LDDT?

Implicative BCS-algebras

- An **implicative BCS-algebra** is an algebra $\langle A; \backslash, 0\rangle$ satisfying the identities and quasi-identity:

$$x \backslash x \approx \mathbf{0}$$
$$(x \backslash y) \backslash z \approx (x \backslash z) \backslash y$$
$$(x \backslash y) \backslash z \approx (x \backslash z) \backslash (y \backslash z)$$
$$x \backslash (y \backslash x) \approx x.$$

  - We denote the class of all implicative BCS-algebra by **iBCS**.

Implicative BCS-algebras are fundamental to the study of binary discriminator varieties.

Implicative BCS-algebras

- **Theorem.** Let **A** be a BCS-algebra. Then **A** has an implicative BCS-algebra term reduct $\langle A; \backslash, 0\rangle$, where $a \backslash b := a - \big((a - (a - b)) - (b - a)\big)$ for all $a, b \in A$.
- **Corollary** (Guzmán). Let **A** be a positive implicative BCK-algebra. Then **A** has an implicative BCK-algebra term reduct $\langle A; \backslash, 0\rangle$, where $a \backslash b := a - \big((a - (a - b)) - (b - a)\big)$ for all $a, b \in A$.
- **Corollary.** Every n+1-potent BCK-algebra **A** has an implicative BCS-algebra term reduct, where $\langle A; \backslash, 0\rangle$, where $a \backslash b := a - \big((a - (a - b)) - (b - a)\big)$ for all $a, b \in A$ and $a - b := ab^n$ for all $a, b \in A$.

An order on implicative BCS-algebras

- **Proposition.** For any implicative BCS-algebra **A**, let $<<$ be the relation defined $\forall\, a, b \in A$ by $a << b$ iff $b \wedge a = a$, where $c \wedge d := c \setminus (c \setminus d)$ $\forall\, c, d \in A$. Then
  - (i) $<<$ is a partial order on A.
  - (ii) If $a << b$ then $a \setminus c << b \setminus c$.
  - (iii) If $a << b$ then $c \setminus b << c \setminus a$.
  - (iv) $0 << a$.
  - (v) For each $a \in A$, the principal $<<$-order ideal $(a]$ is a Boolean lattice.

Thus subclasses of **BCK** can have additional order structure.

Equationally definable partial orders

- Let **A** be an algebra and $E := \{p_i(x, y) \approx q_i(x, y) : i \in I\}$ be a set of equations such that the binary relation $\leq$ defined on A by

$$a \leq b \text{ iff } p_i^{\mathbf{A}}(a, b) = q_i^{\mathbf{A}}(a, b) \qquad i \in I$$

  is a partial ordering. $\leq$ is called an **equationally defined order** for **A**.

- If E defines an order $\leq$ on every algebra **A** in a class **K** then $\leq$ is called an **equationally defined order** for **K**.

- For example, $x \wedge y \approx x$ is an equationally defined order for lattices.

Locally Boolean classes

- An algebra **A** is said to be **locally Boolean** with respect to an equationally defined order $\leq$ for **A** if for each $a \in A$, the principal order ideal $\{b \leq a: b \in A\}$ is a Boolean lattice.
- A quasivariety **K** is called **locally Boolean** if there exists an equationally definable partial order $\leq$ for **K**, such that each **A** $\in$ **K** is locally Boolean with respect to $\leq$.
- **Proposition.** Every variety of n+1-potent BCK-algebras **V** is locally Boolean with respect to the implicative BCS-algebra ordering $\ll$ definable on the members of **V**.
- **Corollary.** Every finite BCK-algebra is locally Boolean.

(Aside: These terms are horrible!)

- ** KEPT (pick-wt=681): 24 [copy,23,demod,22,22] x^y= (x* ((((x* ((x* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x))))* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))))* ((x* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x))))* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))))* ((((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))*x)*x))* ((((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))*x)*x)))* ((((x* ((x* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x))))* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))))* ((x* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x))))* ((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))))* ((((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))*x)*x))* ((((x* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))* ((((x* ((x*y)*y))* ((x*y)*y))* ((y*x)*x))* ((y*x)*x)))*x)*x)).

This is the definition of $x \wedge y$ for 3-potent BCK-algebras!

Right ideal commutative BCK-algebras

- Let **A** be a BCK-algebra. The **principal right ideal** generated by $a \in A$ is the set $\{ab \colon b \in A\}$.

  $\{ab \colon b \in A\}$ is always a subalgebra of **A**.

- **A** is **right ideal commutative** if the Guzmán ordering $\leq^1$ and the BCK-ordering $\leq^0$ coincide on the principal right ideal aA for each $a \in A$. Examples include:
  - All commutative BCK-algebras.
  - All positive implicative BCK-algebras.
  - All $\langle \backslash, 0 \rangle$-subreducts of hoops.

- **Theorem.** The class of all right ideal commutative BCK-algebras is a relative subvariety of **BCK**, but it is not equationally definable.

The terms j_n, $n \geq 1$

- For $n \geq 1$, define the terms

$$j_{-1}(x, y) = x$$
$$j_{2n}(x, y) = y(y(j_{2n-1}(x, y)))$$
$$j_{2n+1}(x, y) = x(x(j_{2n}(x, y))).$$

- Consider the identity

$$j_n(x, y) \approx j_n(y, x). \qquad (\mathrm{J}_n)$$

(The identity (J_n) generalises Cornish's Condition (J).)

Some conjectures....

- **Conjecture.** Let **A** be a right ideal commutative BCK-algebra. If **A** $\models (E_n)$ for $n \geq 1$, then **A** $\models (J_n)$.
- For any $n \geq 1$, define the term $x \wedge^n y = j_n(x, y)$.
- **Conjecture.** Let **A** be an n+1-potent right ideal commutative BCK-algebra. Then $\langle A; \leq^1 \rangle$ is a meet semilattice with 0 and $\forall\, a, b \in A$, glb $\{a, b\} = a \wedge^n b$.
- For any $n \geq 1$, define the term $x \backslash^n y = x - (x \wedge^n y)$.
- **Conjecture.** Let **A** be an n+1-potent right ideal commutative BCK-algebra. Then the term reduct $\langle A; \wedge^n, \backslash^n, 0 \rangle$ is a locally pseudocomplemented meet semilattice.

...and a question

- **Question.** Which subclasses of BCK-algebras are completely determined by the family of ordering relations $\{\leq^n : 0 \leq n \in \omega\} \cup \{\subseteq^2\}$?
- The answer is unknown, but any such class **K** should be a (proper) subclass of some class of n+1-potent right ideal commutative BCK-algebras. For any $\mathbf{A} \in \mathbf{K}$, the BCK operation would then be recovered from the order structure on **A** by setting

$$ab = (a \wedge^{i+1} b)^*_{(a \wedge^i b]}$$

$\forall\ a, b \in A$, where $(a \wedge^{i+1} b)^*_{(a \wedge^i b]}$ denotes the pseudocomplement of $(a \wedge^{i+1} b)$ in the principal $\leq^1$-order ideal $(a \wedge^i b]$.

Some concluding remarks...

- Otter plays an indispensible role in this study, because of the complexity of the terms in question.
- Although the language of BCK-algebras is very restrictive, Otter does not manage the complexity of the terms as well as I would like. What can be done to improve this?
- The proofs are all inductive, and are obtained by hand on disassembling Otter proofs for individual cases. Can we get Otter to efficiently prove things by induction?
- It would be of great assistance in this study if Otter could prove theorems over restricted domains (for example, of 4 or 5 elements). Can this be implemented?