On the assertional logic of the generic double-pointed discriminator variety

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Discriminator varieties

- The *(ternary) discriminator* on a set A is the function $t : A^3 \rightarrow A$ defined $\forall a, b, c \in A$ by

$$t(a, b, c) := \begin{cases} c \text{ if } a = b \\ a \text{ otherwise.} \end{cases}$$

- A *discriminator variety* is a variety $\boldsymbol{V}$ for which there exists a term $t(x, y, z)$ that realises the ternary discriminator on every subdirectly irreducible member of $\boldsymbol{V}$.
- A *double-pointed discriminator variety* is a discriminator variety with two residually distinct constant terms.
- Discriminator varieties enjoy very strong structural properties.
  - Every algebra is $\cong$ a Boolean product of simple algebras.
- Natural examples of discriminator varieties abound.
  - Boolean algebras, relation algebras, monadic algebras, n-valued Post algebras etc.

Most natural examples of discriminator varieties in logic are double-pointed.

Double-pointed discriminator logics

- A *deductive system* is a finitary and structural consequence relation |–.
- A deductive system **S** is a *double-pointed discriminator logic* if it is algebraisable in the sense of Blok and Pigozzi and its equivalent quasivariety is a double-pointed discriminator variety.

  This means ∃ a class **K** of algebras that is to **S** just what **BA** is to **CPC**.
- Double-pointed discriminator logics abound in the literature and include
  - Classical Propositional Logic (**CPC**)
  - Modal logics
    - **S5**, the tetravalent modal logic of Font and Rius
  - Many-valued logics
    - The *n*-dimensional cylindric logics, the *n*-valued Łukasiewicz and Post logics
  - Fuzzy logics
    - Basic fuzzy logics with Baaz delta
  - Logics of vagueness and rough approximation theory
    - 3-valued constructive logic with strong negation, Heyting-Wajsberg logic

'Extracting' a logic from a quasivariety

- For a quasivariety **K** with a constant term **1**, the **1**-*assertional logic* of **K** is the deductive system $S(\mathbf{K}, \mathbf{1})$ determined by the equivalence

  $$\Gamma \vdash_{S(\mathbf{K},1)} \varphi \text{ if and only if } \{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathbf{K}} \varphi \approx \mathbf{1}.$$

  - For example, **CPC** is the **1**-assertional logic of **BA**.

- **Theorem** (Czelakowski, Pigozzi). Let Λ be a language type with a constant term **1**.

  (1) Let **S** be a regularly algebraisable deductive system over Λ. Then **S** is the **1**-assertional logic of a unique relatively **1**-regular quasivariety, namely, its own equivalent quasivariety. In symbols, $\mathbf{S} = S(\text{Alg Mod}^*\ \mathbf{S}, \mathbf{1})$.

  (2) Let **K** be a relatively **1**-regular quasivariety over Λ. Then **K** is the equivalent quasivariety of a unique (Hilbert-style) deductive system, namely, its own assertional logic. In symbols, $\mathbf{K} = \text{Alg Mod}^*\ S(\mathbf{K}, \mathbf{1})$.

So regularly algebraisable logics are coextensive with assertional logics of relatively point regular quasivarieties.

The generic double-pointed discriminator variety

- **Theorem** (McKenzie, B.). Let V be a discriminator variety with discriminator term $t(x, y, z)$. For any algebra $\mathbf{A} \in$ V, the term reduct $\langle A; t^{\mathbf{A}} \rangle$ determines the congruences on $\mathbf{A}$ in the sense that Con $\mathbf{A}$ = Con $\langle A; t^{\mathbf{A}} \rangle$.

- The *generic double-pointed discriminator variety* is the variety $\mathbf{GD}_{01}$ generated by the class of all algebras $\langle A; t, 0, 1 \rangle$ of type $\langle 3, 0, 0 \rangle$, where t is the ternary discriminator on A and 0 and 1 are residually distinct nullary operations.

- Let F be a language type. The *expansion of* $\mathbf{GD}_{01}$ *by F-compatible operations* is the variety of all algebras $\mathbf{A} := \langle A; t, 0, 1, f^{\mathbf{A}} \rangle_{f \in F}$ such that $\langle A; t, 0, 1 \rangle \in \mathbf{GD}_{01}$ and Con $\mathbf{A}$ = Con $\langle A; t, 0, 1 \rangle$.

- **Theorem** (Burris). A variety $\mathbf{V}$ over a language type F is a double-pointed discriminator variety iff it is term equivalent to a subvariety of $\mathbf{GD}_{01}[F]$, the expansion of $\mathbf{GD}_{01}$ by F-compatible operations.

Discriminator logics are determined by $S(\mathbf{GD}_{01}, \mathbf{1})$

- **Proposition.** Let **K** be a relatively **1**-regular quasivariety with **1**-assertional logic **S** := S(**K**, **1**). For any set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathbf{S}} \varphi$ iff there exists some finite set of formulas $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ such that $\forall \mathbf{A} \in \mathbf{K}$ and $\bar{a} \in A^\omega$,

$$\varphi^{\mathbf{A}}(\bar{a}) \equiv \mathbf{1}^{\mathbf{A}} \ (\mathrm{mod}\ \Theta_{\mathbf{K}}^{\mathbf{A}}(\{\langle \varphi_i^{\mathbf{A}}(\bar{a}), \mathbf{1}^{\mathbf{A}}\rangle : i = 1, \ldots, n\})).$$

  - Since every double-pointed discriminator variety is *fully* congruence regular, the proposition applies.

The proposition implies that *every deductive system that is algebraisable, and whose equivalent quasivariety is a double-pointed discriminator variety, is completely determined by its discriminator term*.

For double-pointed discriminator varieties, this implies the study of logics inherent in such varieties reduces to the study of the **1**-assertional logic of $\mathbf{GD}_{01}$.

Extracting a 'Boolean'-like class using the discriminator

- Let A be a set with $1 \in A$. Define operations $\wedge, \vee, \to, \Rightarrow$ $\forall a, b \in A$ by

$$a \to b := \begin{cases} 1 \text{ if } a \neq 1 \\ b \text{ otherwise} \end{cases} \qquad a \wedge b := \begin{cases} b \text{ if } a = 1 \\ a \text{ otherwise} \end{cases}$$

$$a \Rightarrow b := \begin{cases} 1 \text{ if } a = b \\ b \text{ otherwise} \end{cases} \qquad \mathrm{a} \vee \mathrm{b} := \begin{cases} 1 \text{ if } a = 1 \\ b \text{ otherwise} \end{cases}$$

- The Boolean like operations $\wedge, \vee, \Rightarrow$, and $\to$ can be recovered using $t(a, b, c)$ by

$$a \wedge b := t(a, 1, b),\ \ a \vee b := t(a, \mathrm{t}(a, 1, b), b),\ \ a \Rightarrow b := t(b, a, 1),\ \ a \to b := t(1, a, b).$$

- The discriminator can be recovered using $\wedge, \vee, \to$, and $\Rightarrow$ by

$$t(a, b, c) := ((a \Rightarrow b) \to (b \Rightarrow a) \to c) \wedge (b \Rightarrow a).$$

$(\mathbf{SB}^{\cup}\mathbf{A})_{\mathrm{F}}$ denotes the class of all algebras $\langle \mathrm{A}; \wedge, \vee, \Rightarrow, \to, 0, 1\rangle$.

An equational description of $\mathbf{SB}^{\cup}\mathbf{A}$

- An *implication algebra* $\langle A; \Rightarrow, 1\rangle$ is a $\langle \Rightarrow, 1\rangle$-subreduct of a Boolean algebra $\langle A; \cap, \cup, \supset, \sim, 0, 1\rangle$.
  - Implication algebras form the equivalent quasivariety of the $\{\supset\}$-fragment of **CPC**.

- An *implicative BCS-algebra* $\langle A; \to, 1\rangle$ is a $\langle \to, 1\rangle$-subreduct of a *dually* pseudocomplemented semilattice $\langle A; \cup, ^*, 1\rangle$, where $a \to b := a^* \cup b \; \forall a, b \in A$.
  - Implicative BCS-algebras are a 'non-commutative' analogue of implication algebras.

- A *double-pointed implicative BCSK-algebra* is an algebra $\mathbf{A} := \langle A; \Rightarrow, \to, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ where
  - $\langle A; \Rightarrow, 1\rangle$ is an implication algebra
  - $\langle A; \to, 1\rangle$ is an implicative BCS-algebra
  - A $\models x \Rightarrow (y \to x) \approx \mathbf{1}$

    $((x \Rightarrow y) \to y) \to y \approx x \Rightarrow y$

    $\mathbf{0} \to x \approx \mathbf{1}$.

An equational description of $\mathbf{SB}^{\cup}\mathbf{A}$

- A *skew lattice* is an algebra $\mathbf{A} := \langle A; \wedge, \vee\rangle$ of type $\langle 2, 2\rangle$ where
  - $\langle A; \wedge\rangle$ and $\langle A; \vee\rangle$ are bands
  - $\mathbf{A} \models x \wedge (x \vee y) \approx x \approx x \vee (x \wedge y)$
    $(y \vee x) \wedge x \approx x \approx (y \wedge x) \vee x.$
  - Skew lattices arise naturally as bands of idempotents in rings.
- A *double-pointed skew Boolean $\cup$-algebra* ($\mathbf{SB}^{\cup}\mathbf{A}$) is an algebra $\mathbf{A} := \langle A; \wedge, \vee, \Rightarrow, \rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$, where
  - $\langle A; \wedge, \vee\rangle$ is a symmetric skew lattice
  - $\langle A; \Rightarrow, \rightarrow, 0, 1\rangle$ is a double-pointed implicative BCSK-algebra
  - $\mathbf{A} \models x \vee y \approx (x \rightarrow y) \rightarrow y$.
- **Lemma.** The class $\mathbf{SB}^{\cup}\mathbf{A}$ is a variety.
- **Theorem** (B., Leech). $\mathbf{A} \in \mathbf{SB}^{\cup}\mathbf{A}$ is subdirectly irreducible iff $\mathbf{A} \in (\mathbf{SB}^{\cup}\mathbf{A})_F$.
- **Corollary.** $\mathbf{SB}^{\cup}\mathbf{A} = \underline{\mathrm{HSP}}((\mathbf{SB}^{\cup}\mathbf{A})_F)$.

Term equivalence of $\mathbf{SB^{\cup}A}$ and $\mathbf{GD_{01}}$

- **Theorem** (B., Leech).
  (1) Let $\mathbf{A} \in \mathbf{SB^{\cup}A}$. $\forall a, b, c \in A$, let

  $$t(a, b, c) := ((a \Rightarrow b) \to (b \Rightarrow a) \to c) \wedge (b \Rightarrow a).$$

  Then $\mathbf{A}^P := \langle A; t, 0, 1\rangle \in \mathbf{GD_{01}}$.

  (2) Let $\mathbf{A} \in \mathbf{GD_{01}}$. $\forall a, b \in A$, let

  $$a \wedge b := t(a, 1, b)$$
  $$a \vee b := t(a, \mathrm{t}(a, 1, b), b)$$
  $$a \Rightarrow b := t(b, a, 1)$$
  $$a \to b := t(1, a, b).$$

  Then $\mathbf{A}^S := \langle A; \wedge, \vee, \Rightarrow, \to, 0, 1\rangle \in \mathbf{SB^{\cup}A}$.

  (3) Let $\mathbf{A} \in \mathbf{SB^{\cup}A}$. Then $(\mathbf{A}^P)^S = \mathbf{A}$.

  (4) Let $\mathbf{A} \in \mathbf{GD_{01}}$. Then $(\mathbf{A}^S)^P = \mathbf{A}$.

  Hence $\mathbf{GD_{01}}$ and $\mathbf{SB^{\cup}A}$ are term equivalent.

The Skew Boolean Propositional Calculus (**SBPC**)

- Connectives
  - Binary: $\wedge$, $\vee$, $\Rightarrow$, $\rightarrow$
  - Nullary: $\mathbf{0}$
- Axioms

$p \Rightarrow (q \Rightarrow p)$

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

$p \Rightarrow (q \rightarrow p)$

$(p \rightarrow (q \rightarrow r)) \Rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

$((p \rightarrow q) \rightarrow p) \Rightarrow p$

$((p \Rightarrow q) \rightarrow q) \Rightarrow ((q \Rightarrow p) \rightarrow p)$

$(p \Rightarrow q) \rightarrow (p \rightarrow q)$

$p \rightarrow (p \vee q)$

$q \Rightarrow (p \vee q)$

$(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r))$

$(p \wedge q) \Rightarrow p$

$(p \wedge q) \rightarrow q$

$(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r)))$

$\mathbf{0} \rightarrow p$

- Rule of inference

MP$_{\rightarrow}$: $p,\ p \rightarrow q \mid\!-_{\mathbf{SBPC}}\ q$

CPC is the axiomatic extension of **SBPC** by the axiom $(p \rightarrow q) \Rightarrow (p \Rightarrow q)$.

Algebraisability of **SBPC**

- **Lemma** (MP$_{\Rightarrow}$). For all **SBPC**-formulas φ, ψ,

  $$\varphi, \varphi \Rightarrow \psi \vdash_{\mathbf{SBPC}} \psi.$$

- **Proposition.** For any set of **CPC**$^{\supset}$-formulas $\Gamma \cup \{\varphi\}$,

  $$\Gamma \vdash_{\mathbf{CPC}^{\supset}} \varphi \text{ if and only if } \Gamma[\Rightarrow] \vdash_{\mathbf{SBPC}} \varphi[\Rightarrow]$$

  $$\Gamma \vdash_{\mathbf{CPC}^{\supset}} \varphi \text{ if and only if } \Gamma[\rightarrow] \vdash_{\mathbf{SBPC}} \varphi[\rightarrow].$$

- **Theorem** (Deduction Theorem). For any set of **SBPC**-formulas $\Gamma \cup \{\varphi, \psi\}$,

  $$\Gamma, \varphi \vdash_{\mathbf{SBPC}} \psi \text{ if and only if } \Gamma \vdash_{\mathbf{SBPC}} \varphi \rightarrow \psi.$$

- **Theorem. SBPC** is regularly algebraisable with equivalence formulas $\{p \Rightarrow q, q \Rightarrow p\}$ and defining equation $p \approx p \Rightarrow p$.

- **Theorem. SB$^{\cup}$A** is the equivalent quasivariety of **SBPC**.

- **Corollary. SBPC** $= S(\mathbf{GD}_{01}, \mathbf{1})$.

SBPC is 'non-Fregean **CPC**'

- A deductive system **S** is *Fregean* if the relativised *T*-theory interderivability relation $\dashv\vdash_{\mathbf{S}}$ is a congruence on **Fm**
  - A double-pointed discriminator logic is Fregean iff it is definitionally equivalent to an axiomatic expansion of **CPC** by extensional logical connectives.

Informally, **SBPC** is '**CPC** – Fregeanness'

- By dropping Fregeanness from **CPC**,
- Implication bifurcates:
  - $\Leftrightarrow$ is *identity-as-meaning*
  - $\leftrightarrow$ is the *biconditional*
  - For any valuation *v*, $v(\varphi \Leftrightarrow \psi) = 1$ iff $v(\varphi) = v(\psi)$ $v(\varphi \leftrightarrow \psi) = 1$ iff $v(\varphi) \equiv v(\psi)$.
- $\wedge$ and $\vee$ are commutative in **SBPC**, but non-commutative in $\mathbf{SB}^{\cup}\mathbf{A}$. I.e.,
  - $p \wedge q \dashv\vdash_{\mathbf{SBPC}} q \wedge p$, but $p \wedge q \dashv\not\vdash_{\mathbf{SB\cup A}} q \wedge p$
  - $p \vee q \dashv\vdash_{\mathbf{SBPC}} q \vee p$, but $p \vee q \dashv\not\vdash_{\mathbf{SB\cup A}} q \vee p$.

$\mathbf{SB}^{\cup}\mathbf{A}$-algebras with compatible operations

- For an n-ary basic operation f of an algebra $\mathbf{A}$, the *slice* $\underline{f}_i$ of f is the unary operation obtained by assigning fixed values to all but one of the variables, *viz*.

$$\underline{f}_i(b) := f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

- Let F be a language type. Let $\mathbf{SB}^{\cup}\mathbf{A}[F]$ be the variety of algebras with language type $\Lambda[\mathbf{SB}^{\cup}\mathbf{A}] \cup F$ axiomatised by identities axiomatising $\mathbf{SB}^{\cup}\mathbf{A}$ together with the identities:

$$(x \Rightarrow y) \rightarrow ((y \Rightarrow x) \rightarrow (\underline{f}_i(x) \Rightarrow \underline{f}_i(y))) \approx \mathbf{1}.$$

  $\mathbf{SB}^{\cup}\mathbf{A}[F]$ is called the *expansion of* $\mathbf{SB}^{\cup}\mathbf{A}$ *by F-compatible operations.*

- **Theorem.** A variety $\mathbf{V}$ over a language type F is a double-pointed discriminator variety iff it is term equivalent to a subvariety of the expansion of $\mathbf{SB}^{\cup}\mathbf{A}[F]$ by F-compatible operations.

SBPC with compatible operations

- For an *n*-ary connective *f* of **Fm**, the *slice* $\underline{f}_i$ of *f* is the unary operation obtained by assigning fixed values to all but one of the variables, *viz.*

$$\underline{f}_i(\psi) := f(\varphi_1, \ldots, \varphi_{i-1}, \psi, \varphi_{i+1}, \ldots, \varphi_n).$$

- Let *F* be a language type. Let **SBPC**[*F*] be the expansion of **SBPC** to the language type Λ[**SBPC**] $\cup$ *F* obtained by adjoining the axioms

$$(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow (\underline{f}_i(p) \Rightarrow \underline{f}_i(q)))$$

  to an axiomatisation of **SBPC**. **SBPC**[*F*] is called *the expansion of* **SBPC** *by F-extensional connectives*.

- **Theorem.** A deductive system **S** over a language type *F* is a double-pointed discriminator logic iff it is definitionally equivalent to an axiomatic extension of the expansion of **SBPC** by *F*-extensional connectives.

Example: **S5** as a pointed discriminator logic

- Connectives:
  - Binary: $\wedge$, $\vee$, $\Rightarrow$, $\rightarrow$, $\cap$, $\cup$, $\supset$
  - Unary: $\sim$
  - Nullary: **0**, **1**

- Axioms (**SBPC**)

$p \Rightarrow (q \Rightarrow p)$

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

$p \Rightarrow (q \rightarrow p)$

$(p \rightarrow (q \rightarrow r)) \Rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

$((p \rightarrow q) \rightarrow p) \Rightarrow p$

$(p \Rightarrow q) \rightarrow q) \Rightarrow ((q \Rightarrow p) \rightarrow p)$

$(p \Rightarrow q) \rightarrow (p \rightarrow q)$

$p \rightarrow (p \vee q)$

$q \Rightarrow (p \vee q)$

$(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r))$

$(p \wedge q) \Rightarrow p$

$(p \wedge q) \rightarrow q$

$(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r)))$

$\mathbf{0} \rightarrow p$

Example: **S5** as a pointed discriminator logic

- Axioms (**CPC**)

  $p \supset (q \supset p)$
  $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
  $(\sim q \supset \sim p) \supset (p \supset q)$

  $p \supset (p \cup q)$
  $q \supset (p \cup q)$
  $(p \supset r) \supset ((q \supset r) \supset ((p \cup q) \supset r))$

  $(p \cap q) \supset p$
  $(p \cap q) \supset q$
  $(p \supset q) \supset ((p \supset r) \supset (p \supset (q \cap r)))$

  $\mathbf{0} \supset p$
  $p \supset \mathbf{1}$

- Axioms (Compatibility)

  $(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow (\sim p \Rightarrow \sim q))$

  $(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow ((r \cap q) \Rightarrow (r \cap p)))$
  $(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow ((q \cap r) \Rightarrow (p \cap r)))$

  $(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow ((r \cup q) \Rightarrow (r \cup p)))$
  $(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow ((q \cup r) \Rightarrow (p \cup r)))$

  $(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow ((r \supset q) \Rightarrow (r \supset p)))$
  $(p \Rightarrow q) \rightarrow ((q \Rightarrow p) \rightarrow ((q \supset r) \Rightarrow (p \supset r)))$

- Axioms (Link)

  $((p \supset q) \supset q) \Rightarrow ((q \supset p) \supset p)$
  $(p \supset q) \Rightarrow (p \rightarrow q)$

- Rule of inference

  $\mathrm{MP}_{\rightarrow}$: $p, p \rightarrow q \vdash_{\mathbf{S5}} q$

Term equivalence of monadic and super-Boolean algebras

- A *super-Boolean algebra* is an algebra $\langle A; \cap, \cup, \supset, \wedge, \vee, \Rightarrow, \rightarrow, \sim, 0, 1\rangle$ where
  - $\langle A; \cap, \cup, \supset, \sim, 0, 1\rangle$ is a Boolean algebra
  - $\langle A; \wedge, \vee, \Rightarrow, \rightarrow, 0, 1\rangle$ is a double-pointed skew Boolean $\cup$-algebra, and
  - The Boolean algebra operations are compatible with $\langle A; \wedge, \vee, \Rightarrow, \rightarrow, 0, 1\rangle$.
- **Theorem.**

  (1) Let **A** be a super-Boolean algebra. $\forall a, b \in A$, let $\Box a := (a \rightarrow 0) \Rightarrow 0$.

  Then $\mathbf{A}^M := \langle A; \cap, \cup, \supset, \sim, \Box, 0, 1\rangle$ is a monadic algebra.

  (2) Let **A** be a monadic algebra. $\forall a, b \in A$, let

  $a \wedge b := a \cap (\Box a \supset b)$ $\qquad$ $a \Rightarrow b := \Box(a \equiv b) \cup b$

  $a \vee b := \Box(\Box a \supset b) \supset b$ $\qquad$ $a \rightarrow b := \Box a \supset b$.

  Then $\mathbf{A}^S := \langle A; \cap, \cup, \supset, \wedge, \vee, \Rightarrow, \rightarrow, \sim, 0, 1\rangle$ is a super-Boolean algebra.

  (3) Let **A** be a super-Boolean algebra. Then $(\mathbf{A}^M)^S = \mathbf{A}$.

  (4) Let **A** be a monadic algebra. Then $(\mathbf{A}^S)^M = \mathbf{A}$.

  Hence the varieties of monadic and super-Boolean algebras are term equivalent.

An alternative explanation for term equivalence

- For a class **K** of algebras over a language type Λ, let $\mathbf{K}^t$ denote the class obtained from **K** by adjoining a new ternary function symbol $t(x, y, z)$ to Λ such that the realisation of t on any $\mathbf{A} \in \mathbf{K}$ is the ternary discriminator.
- **Proposition** (Burris, McKenzie, Valeriote). $\mathbf{MA} = \underline{\mathrm{HSP}}(\mathbf{BA}^t)$.

Concluding Remarks

- Every double-pointed discriminator variety **V** over a language type F with **1** has an associated assertional logic $S(\mathbf{GD_{01}}[F]^{+}, \mathbf{1}) \equiv \mathbf{SBPC}[F]^{+}$, such that

$$\text{Alg Mod}^{*}\ S(\mathbf{GD_{01}}[F]^{+}, \mathbf{1}) \equiv \text{Alg Mod}^{*}\ \mathbf{SBPC}[F]^{+} \equiv \mathbf{V}.$$

- The deductive system $S(\mathbf{GD_{01}}, \mathbf{1})$ *qua* **SBPC** has a coherent *meaning*. Informally, **SBPC** is **CPC** minus the Fregean property.

- The main *novelty* of the work is the axiomatisation of $S(\mathbf{GD_{01}}, \mathbf{1})$ *qua* **SBPC**.

- The main *technical insight* of the work is that even in the restricted setting of regularly algebraisable logics with the uniterm DDT, the relationship between

  identity-as-meaning $\Leftrightarrow$ and the biconditional $\leftrightarrow$

  is quite complicated. This warrants further investigation.