Frames vs. Bases

- A set of vectors form a *basis* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$ and are linearly independent.
- A set of $N \geq M$ vectors form a *frame* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$.

Basis Matrix

Let $\mathcal{B}$ consist of the M basis vectors, $\mathbf{b}_1 \ldots \mathbf{b}_N \in \mathbb{R}^M$. Let $\mathbf{x} \in \mathbb{R}^M$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\mathcal{B}$. It follows that

$$\mathbf{y} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_M\mathbf{b}_M.$$

This is just the matrix vector product

$$\mathbf{y} = \mathbf{B}\mathbf{x}$$

where the *basis matrix*, $\mathbf{B}$, is the $M \times M$ matrix,

$$\mathbf{B} = \left[\, \mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_M \,\right].$$

Inverse Basis Matrix

To find the representation of the vector $\mathbf{y}$ in the basis $\mathcal{B}$ we multiply $\mathbf{y}$ by $\mathbf{B}^{-1}$:

$$\mathbf{x} = \mathbf{B}^{-1}\mathbf{y}.$$

The components of the representation of $\mathbf{y}$ in $\mathcal{B}$ are inner products of $\mathbf{y}$ with the rows of $\mathbf{B}^{-1}$. The transposes of these row vectors form a *dual basis* $\tilde{\mathcal{B}}$.

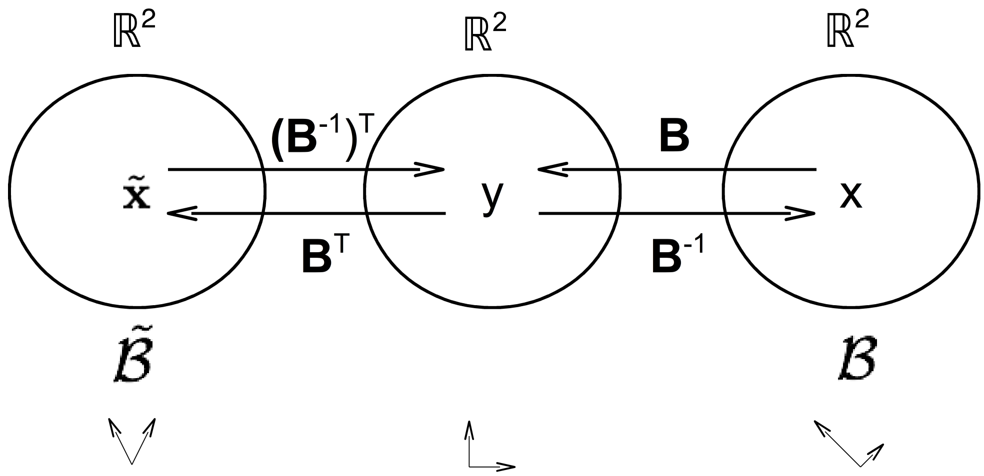

Figure 1: Primal $\mathcal{B}$ (right) and dual $\tilde{\mathcal{B}}$ (left) bases and standard basis (center). The vectors which comprise $\tilde{\mathcal{B}}$ are the transposes of the rows of $\mathbf{B}^{-1}$.

Frame Matrix

Let $\mathcal{F}$ consist of the N frame vectors, $\mathbf{f}_1 \ldots \mathbf{f}_N \in \mathbb{R}^M$, where $N \geq M$. Let $\mathbf{x} \in \mathbb{R}^N$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\mathcal{F}$. It follows that

$$\mathbf{y} = x_1\mathbf{f}_1 + x_2\mathbf{f}_2 + \cdots + x_N\mathbf{f}_N.$$

This is just the matrix vector product

$$\mathbf{y} = \mathbf{F}\mathbf{x}$$

where the *frame matrix*, $\mathbf{F}$, is the $M \times N$ matrix,

$$\mathbf{F} = \left[\; \mathbf{f}_1 \mid \mathbf{f}_2 \mid \;\ldots\; \mid \mathbf{f}_N \;\right].$$

Inverse Frame Matrix (contd.)

We might guess that

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{y}$$

where $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$. Unfortunately, because $\mathbf{F}$ is not square, it has no simple inverse. However, it has an infinite number of *right-inverses*. Each of the $\mathbf{x}$ produced when $\mathbf{y}$ is multiplied by a distinct right-inverse is a distinct representation of the vector $\mathbf{y}$ in the frame, $\mathcal{F}$.

Pseudoinverse

We observe that the *pseudoinverse*

$$\mathbf{F}^{+} = \mathbf{F}^{\mathrm{T}}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^{-1}$$

is a right-inverse of $\mathbf{F}$. We call the $N \times M$ matrix, $\mathbf{F}^{+}$, an *inverse frame matrix* because it maps vectors, $\mathbf{y} \in \mathbb{R}^{M}$, into representations, $\mathbf{x} \in \mathbb{R}^{N}$.

Frame Bounds

Let $\mathcal{F}$ consist of the N frame vectors, $\mathbf{f}_1 \ldots \mathbf{f}_N \in \mathbb{R}^M$, where $N \geq M$, and let $\mathbf{F}^+$ be the inverse frame matrix. $\mathcal{F}$ is a frame iff for all $\mathbf{y} \in \mathbb{R}^M$ there exist A and B where $0 < A \leq B < \infty$ and where

$$\frac{1}{B}||\mathbf{y}||^2 \leq ||\mathbf{F}^+\mathbf{y}||^2 \leq \frac{1}{A}||\mathbf{y}||^2.$$

A and B are called the *frame bounds*.

Dual Frame

If $\mathcal{F}$ consists of the N frame vectors, $\mathbf{f}_1 \ldots \mathbf{f}_N \in \mathbb{R}^M$, with inverse frame matrix $\mathbf{F}^+$, then the *dual frame*, $\widetilde{\mathcal{F}}$, consists of the N frame vectors, $\widetilde{\mathbf{f}}_1 \ldots \widetilde{\mathbf{f}}_N \in \mathbb{R}^M$:

$$\left(\mathbf{F}^+\right)^{\mathrm{T}} = \left[\widetilde{\mathbf{f}}_1 \mid \widetilde{\mathbf{f}}_2 \mid \ \ldots \ \mid \widetilde{\mathbf{f}}_N\right].$$

Let $\widetilde{\mathbf{x}} \in \mathbb{R}^N$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\widetilde{\mathcal{F}}$. It follows that

$$\mathbf{y} = \left(\mathbf{F}^+\right)^{\mathrm{T}} \widetilde{\mathbf{x}}.$$

Consequently, $(\mathbf{F}^+)^{\mathrm{T}}$ is the frame matrix for the dual frame, $\widetilde{\mathcal{F}}$.

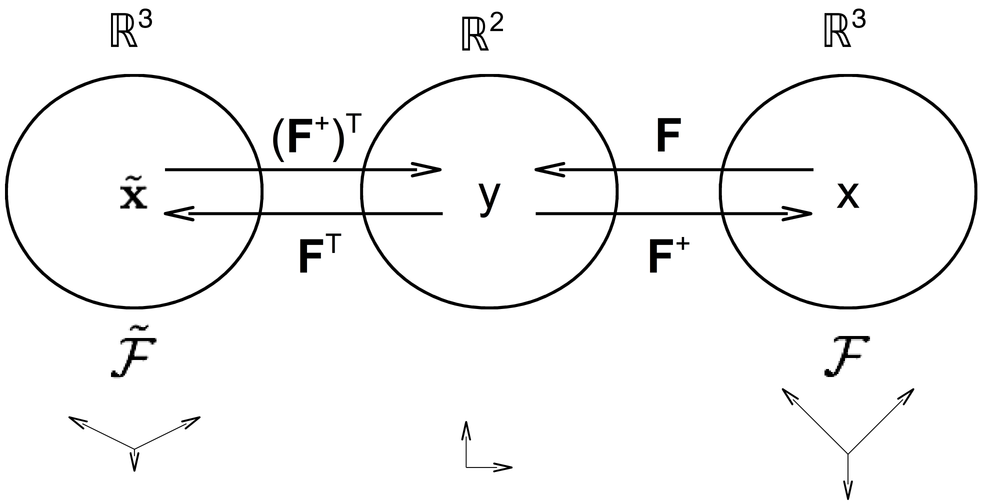

Figure 2: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) frames and standard basis (center). The vectors which comprise $\tilde{\mathcal{F}}$ are the transposes of the rows of $\mathbf{F}^{+}$.

Dual Frame (contd.)

Because $\mathbf{F}^{\mathrm{T}}$ is a right inverse of $(\mathbf{F}^{+})^{\mathrm{T}}$:

$$\left(\mathbf{F}^{+}\right)^{\mathrm{T}}\mathbf{F}^{\mathrm{T}} = \mathbf{I}.$$

It follows that $\mathbf{F}^{\mathrm{T}}$ is the inverse frame matrix for the dual frame, $\widetilde{\mathcal{F}}$, and

$$A||\mathbf{y}||^2 \leq ||\mathbf{F}^{\mathrm{T}}\mathbf{y}||^2 \leq B||\mathbf{y}||^2.$$

for all $\mathbf{y} \in \mathbb{R}^M$.

Example

What is the representation of $\mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$ in the frame formed by the vectors $\mathbf{f}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{f}_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^{\mathrm{T}}$?

$$\mathbf{F} = \begin{bmatrix} 0.70711 & -0.70711 & 0 \\ 0.70711 & 0.70711 & -1 \end{bmatrix}$$

$$\mathbf{F}^{+} = \begin{bmatrix} 0.70711 & 0.35355 \\ -0.70711 & 0.35355 \\ 0 & -0.5 \end{bmatrix}$$

$$\mathbf{F}^{+}\mathbf{y} = \begin{bmatrix} 1.06066 \\ -0.35355 \\ -0.5 \end{bmatrix}$$

Tight-Frames

If $A = B$ then

$$||\mathbf{F}^{\mathrm{T}}\mathbf{y}||^2 = A||\mathbf{y}||^2$$

and $\mathcal{F}$ is said to be a *tight-frame*. When $\mathcal{F}$ is a tight-frame,

$$\mathbf{F}^{+} = \frac{1}{A}\mathbf{F}^{\mathrm{T}}.$$

If $||\mathbf{f}_i|| = 1$ for all frame vectors, $\mathbf{f}_i$, then A equals the overcompleteness of the representation. When $A = B = 1$, then $\mathcal{F}$ is an *orthonormal basis* and $\mathcal{F} = \widetilde{\mathcal{F}}$.

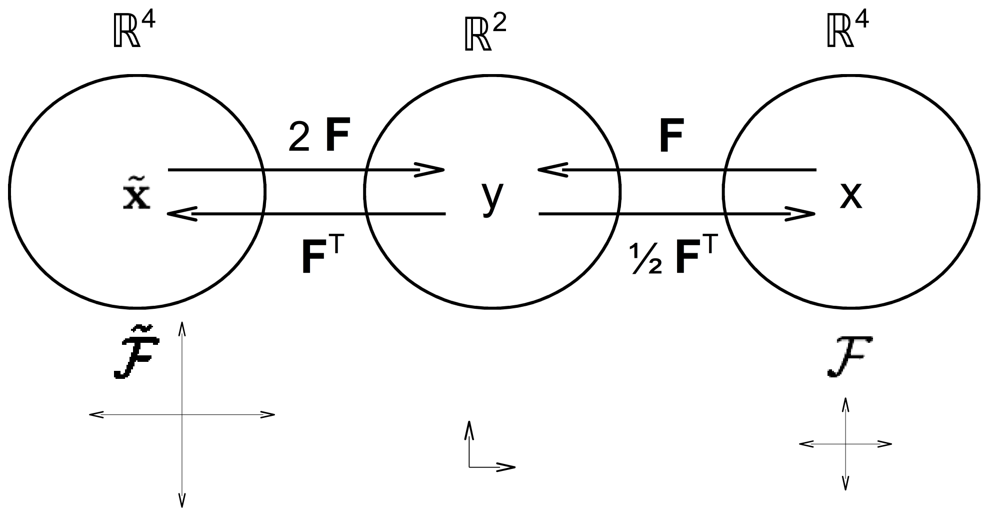

Figure 3: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness two and standard basis (center).

Example

What is the representation of $\mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$ in the frame formed by the vectors $\mathbf{f}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{f}_4 = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}$?

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{F}^{+} = \frac{1}{2}\mathbf{F}^{\mathrm{T}} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \\ 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}$$

$$\frac{1}{2}\mathbf{F}^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

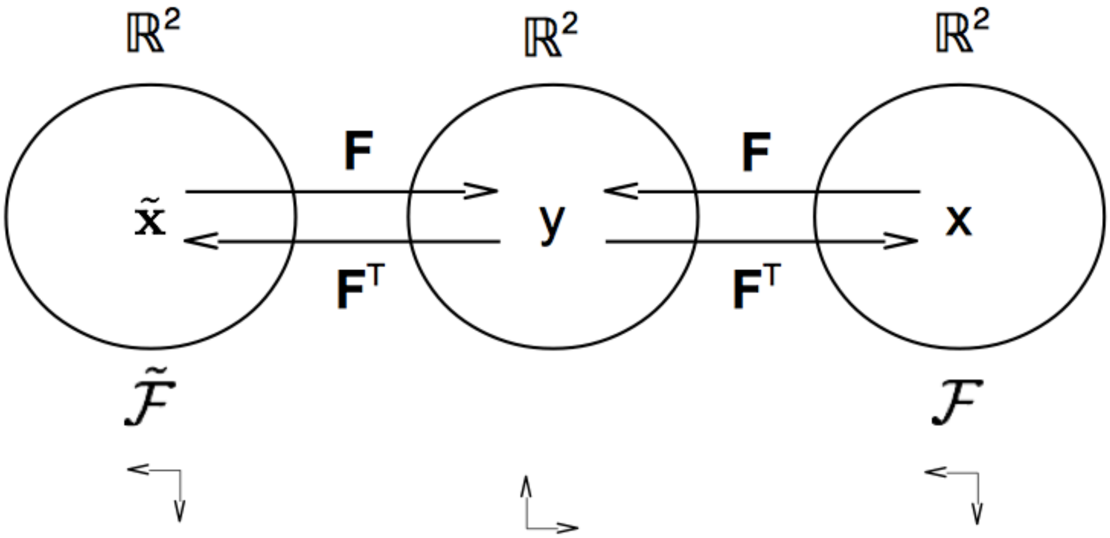

Figure 4: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness one (orthonormal bases) and standard basis (center).

Summary of Notation

- $\mathbf{y} \in \mathbb{R}^M$ – a vector.
- $\mathbf{x} \in \mathbb{R}^N$ – a representation of $\mathbf{y}$ in $\mathcal{F}$.
- $\mathbf{f}_1 \ldots \mathbf{f}_N \in \mathbb{R}^M$ where $N \geq M$ – frame vectors for $\mathcal{F}$.
- $\mathbf{F} = \left[\, \mathbf{f}_1 \mid \mathbf{f}_2 \mid \ \ldots \ \mid \mathbf{f}_N \right]$ – frame matrix for $\mathcal{F}$.
- $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^M$.
- $\mathbf{F}^+ = \mathbf{F}^{\mathrm{T}} \left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)^{-1}$ – inverse frame matrix for $\mathcal{F}$.
- $\mathbf{F}^+ : \mathbb{R}^M \to \mathbb{R}^N$.
- $0 < A \leq B < \infty$ – bounds for $\mathcal{F}$.

Summary of Notation (contd.)

- $\widetilde{\mathbf{x}} \in \mathbb{R}^M$ – a representation of $\mathbf{y}$ in $\widetilde{\mathcal{F}}$.
- $\widetilde{\mathbf{f}}_1 \ldots \widetilde{\mathbf{f}}_N \in \mathbb{R}^M$ – frame vectors for $\widetilde{\mathcal{F}}$.
- $\left(\mathbf{F}^{+}\right)^{\mathrm{T}} = \left[\widetilde{\mathbf{f}}_1 \mid \widetilde{\mathbf{f}}_2 \mid \ \ldots \ \mid \widetilde{\mathbf{f}}_N\right]$ – frame matrix for $\widetilde{\mathcal{F}}$.
- $\left(\mathbf{F}^{+}\right)^{\mathrm{T}} : \mathbb{R}^N \rightarrow \mathbb{R}^M$.
- $\mathbf{F}^{\mathrm{T}}$ – inverse frame matrix for $\widetilde{\mathcal{F}}$.
- $\mathbf{F}^{\mathrm{T}} : \mathbb{R}^M \rightarrow \mathbb{R}^N$.
- $0 < \frac{1}{B} \leq \frac{1}{A} < \infty$ – bounds for $\widetilde{\mathcal{F}}$.