The 2D Fourier Transform

The analysis and synthesis formulas for the 2D continuous Fourier transform are as follows:

- **Analysis**

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\, dy$$

- **Synthesis**

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du\, dv$$

Separability of 2D Fourier Transform

The 2D analysis formula can be written as a 1D analysis in the x direction followed by a 1D analysis in the y direction:

$$F(u,v) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi ux} dx \right] e^{-j2\pi vy} dy.$$

The 2D synthesis formula can be written as a 1D synthesis in the x direction followed by a 1D synthesis in y direction:

$$f(x,y) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(u,v) e^{j2\pi ux} du \right] e^{j2\pi vy} dv.$$

<u>Separability Theorem</u>

$$f(x,y) = f(x)g(y) \xrightarrow{\mathcal{F}} F(u,v) = F(u)G(v)$$

Proof:

$$F(u,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-j2\pi ux}e^{-j2\pi vy}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}\,dx \int_{-\infty}^{\infty} g(y)e^{-j2\pi vy}dy$$

$$= F(u)\,G(v)$$

<u>The 2D Discrete Fourier Transform</u>

The analysis and synthesis formulas for the 2D discrete Fourier transform are as follows:

- **Analysis**

$$\hat{F}(k,\ell) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F(m,n) e^{-j2\pi\left(k\frac{m}{M}+\ell\frac{n}{N}\right)}$$

- **Synthesis**

$$F(m,n) = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{\ell=0}^{N-1} \hat{F}(k,\ell) e^{\,j2\pi\left(k\frac{m}{M}+\ell\frac{n}{N}\right)}$$

Separability of the 2D DFT

$$\hat{F}(k,\ell) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\left[\frac{1}{\sqrt{M}}\sum_{m=0}^{M-1}F(m,n)e^{-j2\pi\left(k\frac{m}{M}\right)}\right]e^{-j2\pi\left(\ell\frac{n}{N}\right)}$$

The 2D forward DFT can be written in matrix notation:

$$\hat{\mathbf{F}} = (\mathbf{W}^*\mathbf{F})\,\mathbf{W}^*$$

where

$$W^*_{mn} = \frac{1}{\sqrt{N}}e^{-j2\pi m\frac{n}{N}}$$

and

$$F(m,n) = \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}\left[\frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}\hat{F}(k,\ell)e^{\,j2\pi\left(k\frac{m}{M}\right)}\right]e^{\,j2\pi\left(\ell\frac{n}{N}\right)}.$$

Separability of the 2D DFT (contd.)

The 2D inverse DFT can be written in matrix notation:

$$\mathbf{F} = \left(\mathbf{W}\hat{\mathbf{F}}\right)\mathbf{W}$$

where

$$W_{mn} = \frac{1}{\sqrt{N}} e^{\,j2\pi m \frac{n}{N}}.$$