Convolution with Highpass Filter

Convolution with the H_1 filter is implemented as multiplication with a circulant matrix:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

where u_i are the components of the upper half-band image and x_i are the components of the input image.

Convolution with Lowpass Filter

Convolution with the H_0 filter is also implemented as multiplication with a circulant matrix:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

where v_i are the components of the lower half-band image and x_i are the components of the input image.

Downsampling the Upper Half-band

The factor of two downsampling of the upper half-band image is accomplished by discarding the even numbered rows. This forms the $\mathbf{U}_1$ matrix:

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

Downsampling the Lower Half-band

The factor of two downsampling of the lower half-band image is accomplished by discarding the even numbered rows. This forms the $\mathbf{L}_1$ matrix:

$$\begin{bmatrix} v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

Two Channel Analysis Matrix

The entire analysis half of the two channel subband transform can be accomplished by combining the $\mathbf{U}_1$ and $\mathbf{L}_1$ matrices into a single matrix:

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}.$$

Written using block matrix notation:

$$\left[\frac{\mathbf{u}'}{\mathbf{v}'}\right] = \frac{1}{\sqrt{2}} \left[\frac{\mathbf{U}_1}{\mathbf{L}_1}\right] \mathbf{x}.$$

Two Channel Synthesis Matrix

Because the Haar transform is unitary, the synthesis matrix is simply the transpose of the analysis matrix:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix}.$$

Written using block matrix notation:

$$\frac{1}{\sqrt{2}} \left[\mathbf{U}_1^{\mathrm{T}} \middle| \mathbf{L}_1^{\mathrm{T}} \right] \frac{1}{\sqrt{2}} \left[\frac{\mathbf{U}_1}{\mathbf{L}_1} \right] = \mathbf{I}_1$$

so that

$$\mathbf{x} = \frac{1}{\sqrt{2}} \left[\mathbf{U}_1^{\mathrm{T}} \middle| \mathbf{L}_1^{\mathrm{T}} \right] \left[\frac{\mathbf{u}'}{\mathbf{v}'} \right].$$

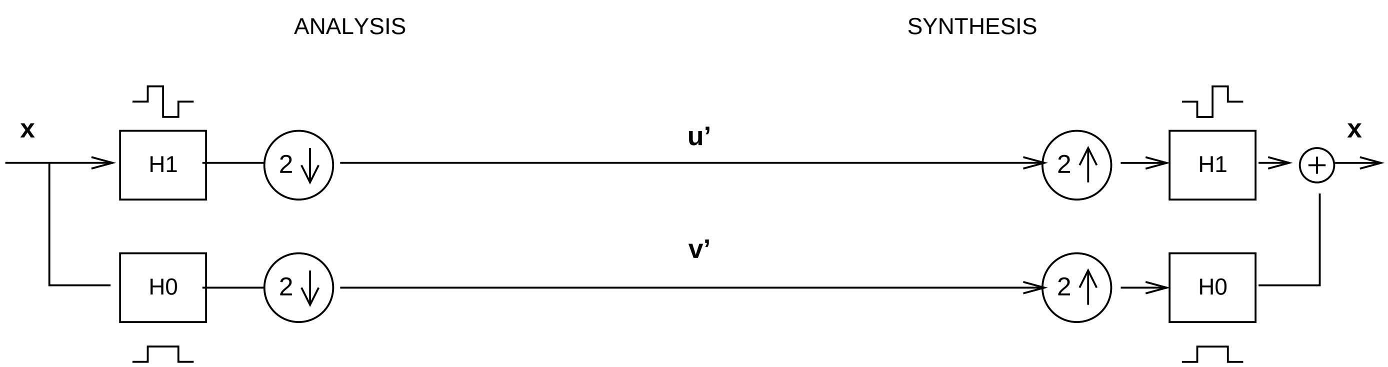

Figure 1: Two channel Haar transform.

Three Channel Analysis Matrix

The two channel subband coding scheme can be extended to a three channel subband coding scheme by using two channel subband coding recursively to encode the lower half-band image:

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ w_1 \\ w_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix}.$$

Three Channel Analysis Matrix

The upper left quadrant of the second stage analysis matrix is the identity matrix. This allows the upper half-band to remain unchanged. However, the lower right quadrant of the second stage analysis matrix is a half-sized copy of the first stage analysis matrix. This encodes the lower half-band. In block matrix notation:

$$\left[\begin{array}{c} \mathbf{u}' \\ \hline \mathbf{v}'' \\ \hline \mathbf{w}'' \end{array}\right] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \sqrt{2}\,\mathbf{I}_2 & \mathbf{0} \\ \hline \mathbf{0} & \begin{array}{c} \mathbf{U}_2 \\ \hline \mathbf{L}_2 \end{array} \end{array}\right] \left[\begin{array}{c} \mathbf{u}' \\ \hline \mathbf{v}' \end{array}\right].$$

Second Stage Synthesis Matrix

The second stage analysis matrix is also orthonormal, so that the second stage analysis can be undone by multiplication by the transpose:

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ w_1 \\ w_3 \end{bmatrix}.$$

Written in block matrix notation:

$$\frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} \sqrt{2}\,\mathbf{I}_2 & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & \mathbf{U}_2^{\mathrm{T}} & \mathbf{L}_2^{\mathrm{T}} \end{array} \right] \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \sqrt{2}\,\mathbf{I}_2 & \mathbf{0} \\ \hline \mathbf{0} & \begin{array}{c} \mathbf{U}_2 \\ \hline \mathbf{L}_2 \end{array} \end{array} \right] = \mathbf{I}_1$$

so that

$$\left[\begin{array}{c} \mathbf{u}' \\ \hline \mathbf{v}' \end{array} \right] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} \sqrt{2}\,\mathbf{I}_2 & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & \mathbf{U}_2^{\mathrm{T}} & \mathbf{L}_2^{\mathrm{T}} \end{array} \right] \left[\begin{array}{c} \mathbf{u}' \\ \hline \mathbf{v}'' \\ \hline \mathbf{w}'' \end{array} \right].$$

Three Channel Analysis Matrix

To perform three channel subband encoding, the input image is multiplied by the first stage analysis matrix and the second stage analysis matrix in succession:

$$\left[\begin{array}{c}\mathbf{u}' \\ \hline \mathbf{v}'' \\ \hline \mathbf{w}''\end{array}\right] = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c}\sqrt{2}\,\mathbf{I}_2 & \mathbf{0} \\ \hline \mathbf{0} & \begin{array}{c}\mathbf{U}_2 \\ \hline \mathbf{L}_2\end{array}\end{array}\right]\frac{1}{\sqrt{2}}\left[\begin{array}{c}\mathbf{U}_1 \\ \hline \mathbf{L}_1\end{array}\right]\mathbf{x}.$$

Three Channel Synthesis Matrix

Three channel subband decoding is accomplished by multiplying the coded image by the transposes of the first and second stage analysis matrices in reverse order:

$$\mathbf{x} = \frac{1}{\sqrt{2}} \left[\, \mathbf{U}_1^{\mathrm{T}} \,\middle|\, \mathbf{L}_1^{\mathrm{T}} \,\right] \frac{1}{\sqrt{2}} \left[\begin{array}{c|cc} \sqrt{2}\,\mathbf{I}_2 & \multicolumn{2}{c}{\mathbf{0}} \\ \hline \mathbf{0} & \mathbf{U}_2^{\mathrm{T}} & \mathbf{L}_2^{\mathrm{T}} \end{array} \right] \left[\begin{array}{c} \mathbf{u}' \\ \hline \mathbf{v}'' \\ \hline \mathbf{w}'' \end{array} \right].$$

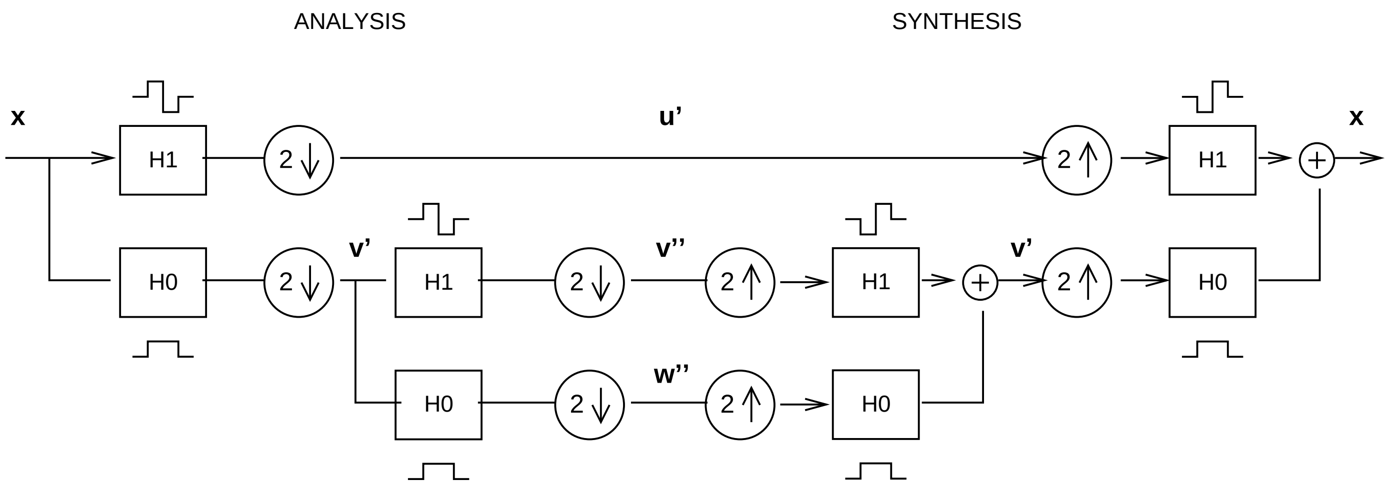

Figure 2: Three channel Haar transform.