

Behavior of the Shah function under Scaling

Recall that the Shah function is defined as follows:

$$\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n)$$

and that the impulse behaves as follows when scaled:

$$\delta(at) = \frac{1}{|a|} \delta(t).$$

Using the above we can derive an expression for a scaled Shah function:

$$\begin{aligned} \text{III}(at) &= \sum_{n=-\infty}^{\infty} \delta(at - n) \\ &= \sum_{n=-\infty}^{\infty} \delta \left[a \left(t - \frac{n}{a} \right) \right] \\ &= \frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta \left(t - \frac{n}{a} \right). \end{aligned}$$

Behavior of the Shah function under Scaling (contd.)

When $a = 1/\tau$ and $\tau > 0$:

$$\text{III}(t/\tau) = \tau \sum_{n=-\infty}^{\infty} \delta(t - n\tau).$$

Dividing both sides by τ results in a Shah function composed of impulses of unit magnitude spaced τ apart:

$$\text{III}(t/\tau) / \tau = \sum_{n=-\infty}^{\infty} \delta(t - n\tau).$$

Fourier Transform of the Shah Function

The Shah function is its own Fourier transform:

$$\mathcal{F} \{ \text{III}(t) \} (s) = \text{III}(s).$$

We can use the Similarity Theorem

$$\mathcal{F} \{ f(at) \} (s) = \frac{1}{|a|} F \left(\frac{s}{a} \right)$$

to derive an expression for the Fourier transform of a Shah function composed of impulses of unit magnitude spaced τ apart. The Fourier transform is

$$\mathcal{F} \{ \text{III}(t/\tau)/\tau \} (s) = \text{III}(\tau s)$$

where

$$\text{III}(\tau s) = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} \delta(s - n/\tau).$$

This is a Shah function composed of impulses of magnitude $\frac{1}{\tau}$ spaced $\frac{1}{\tau}$ apart.

Sampling Theorem

Let f_d be a discretely sampled version of f :

$$\begin{aligned} f_d(t) &= \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\tau) \\ &= f(t) \sum_{n=-\infty}^{\infty} \delta(t - n\tau). \end{aligned}$$

Since

$$\text{III}(t/\tau)/\tau = \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

it follows that:

$$f_d(t) = f(t) \cdot \text{III}(t/\tau) / \tau.$$

Sampling Theorem (contd.)

- **Question** For what values of τ can we recover f from f_d ?
- **Idea** Perhaps it's easier to recover F from F_d . Taking the Fourier transform of both sides of

$$f_d(t) = f(t) \cdot \text{III}(t/\tau) / \tau$$

yields

$$\begin{aligned} \mathcal{F}\{f_d(t)\}(s) &= \mathcal{F}\{f(t)\}(s) * \mathcal{F}\{\text{III}(t/\tau)/\tau\}(s) \\ F_d(s) &= F(s) * \text{III}(\tau s) \\ &= F(s) * \frac{1}{\tau} \sum_{n=-\infty}^{\infty} \delta(s - n/\tau) \\ &= \frac{1}{\tau} \sum_{n=-\infty}^{\infty} F(s) * \delta(s - n/\tau) \\ &= \frac{1}{\tau} \sum_{n=-\infty}^{\infty} F(s - n/\tau). \end{aligned}$$

Sampling Theorem (contd.)

Let $F(s) = 0$ for $|s| \geq s_1$. If $0 < \tau < 1/(2s_1)$, then multiplying both sides by an ideal lowpass filter, $\tau\Pi(\tau s)$, yields:

$$\begin{aligned}\tau\Pi(\tau s) \cdot F_d(s) &= \Pi(\tau s) \sum_{n=-\infty}^{\infty} F(s - n/\tau) \\ &= F(s).\end{aligned}$$

What does this look like in the time domain?
Applying \mathcal{F}^{-1} to both sides of

$$\tau\Pi(\tau s) \cdot F_d(s) = F(s)$$

yields

$$\begin{aligned}\mathcal{F}^{-1}\{\tau\Pi(\tau s)\}(t) * \mathcal{F}^{-1}\{F_d(s)\}(t) &= \mathcal{F}^{-1}\{F(s)\}(t) \\ \sin(\pi t/\tau)/(\pi t/\tau) * f_d(t) &= f(t).\end{aligned}$$

Useful jargon: $\sin(\pi t/\tau)/(\pi t/\tau)$ is called the *sinc* function, s_1 is termed the *folding frequency*, and $\tau = 1/(2s_1)$ is termed the *Nyquist rate*.

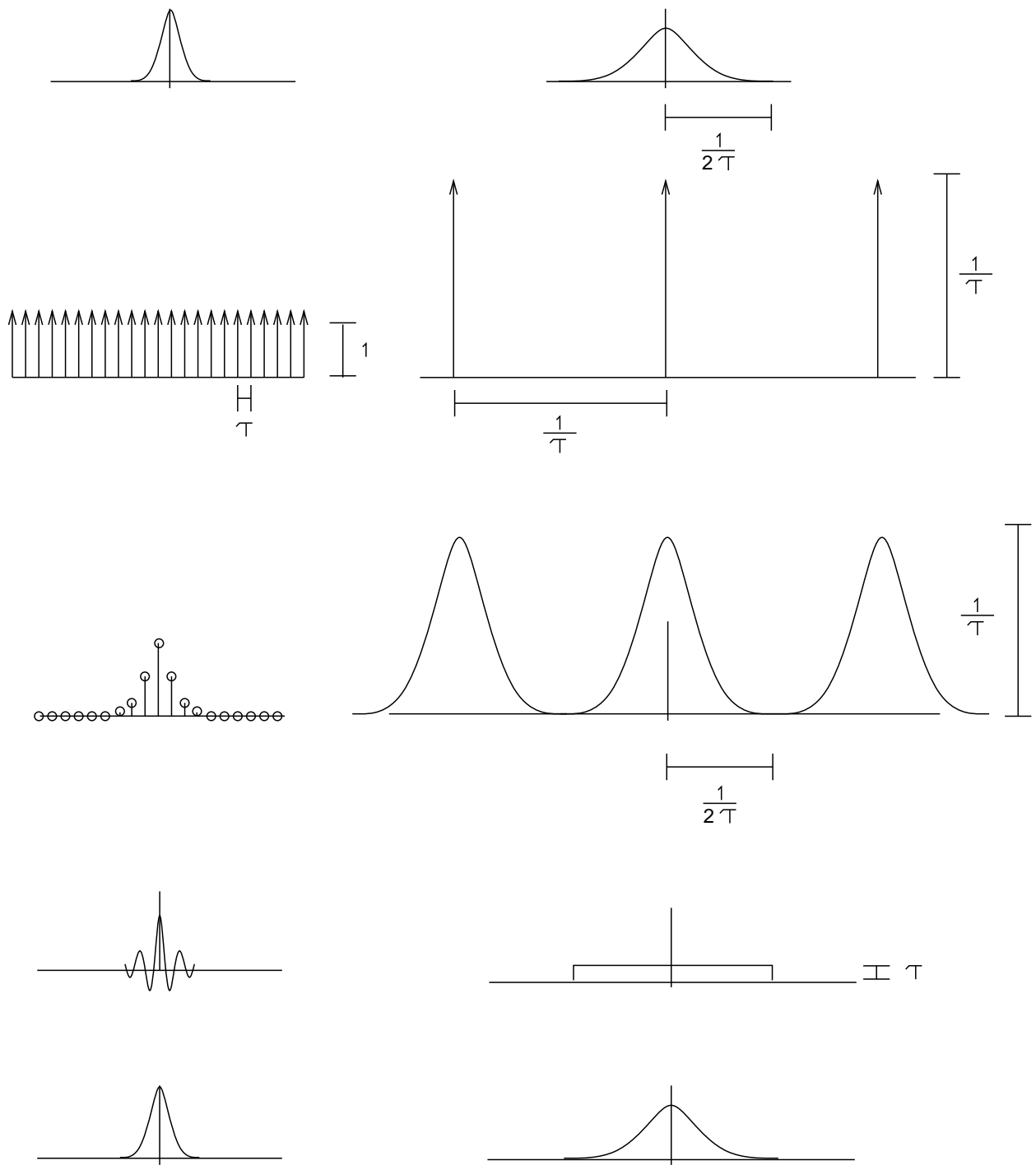


Figure 1: Reconstructing a continuous signal from discrete samples. Time domain (left). Frequency domain (right).

Two Channel Subband Coding

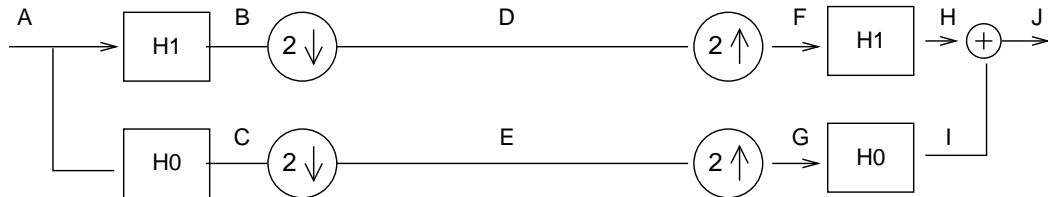


Figure 2: Two channel subband coding.

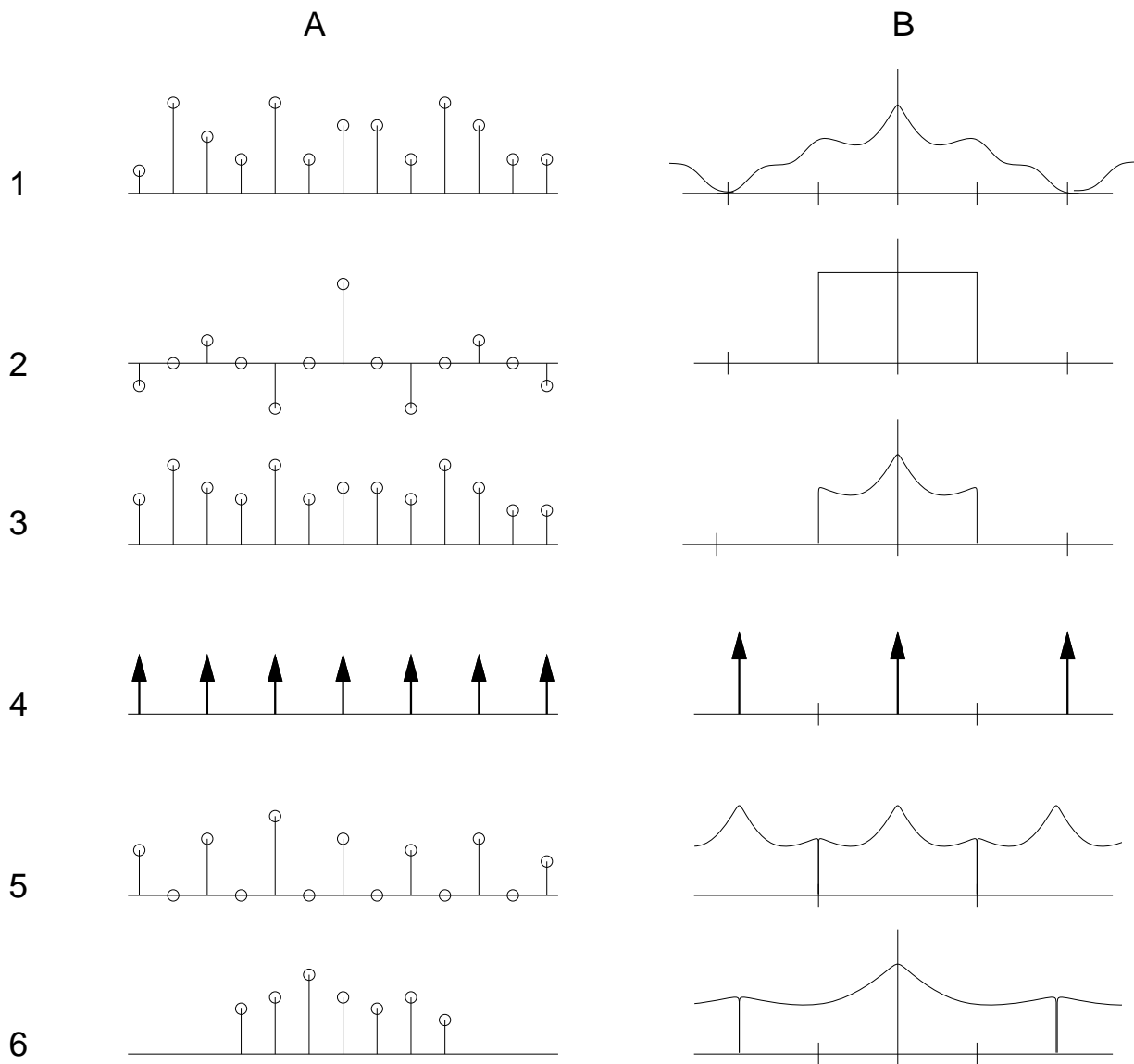


Figure 3: Lower halfband.

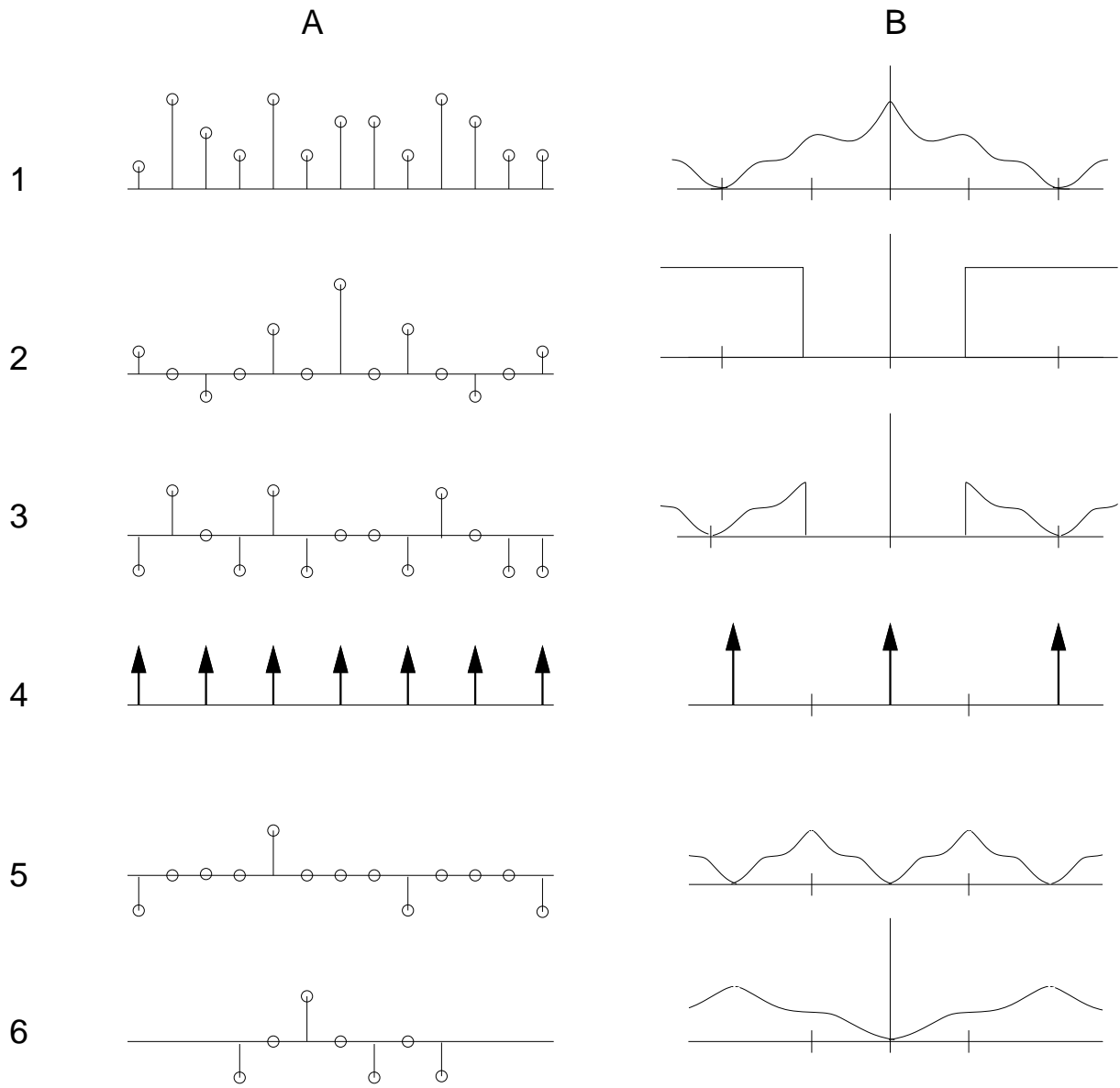


Figure 4: Upper halfband.