CS 530: Geometric and Probabilistic Methods in Computer Science
Homework 6 (Fall ’13)

1. Let \( f(t) = e^{-\pi t^2}, \quad f''(t) = 2\pi e^{-\pi t^2}(2\pi t^2 - 1), \) and \( g(t) = at + b. \) Prove or disprove the following:
\[ \langle f'', g \rangle = 0 \]
for all \( a \) and \( b. \)

2. Let \( \Psi(t) = e^{-\pi t^2} \cos(2\pi s_0 t) \) and \( f(t) = e^{j2\pi s_1 t}. \) Give an expression for \( F(a, b) \), the continuous wavelet transform of \( f(t). \)

3. The \( n \)-th moment of \( \Psi \) is defined to be \( M_n\{\Psi\} = \int_{-\infty}^{\infty} t^n \Psi(t)dt. \) Let \( f(t) = e^{-\pi t^2}, \quad f'(t) = -2\pi t e^{-\pi t^2}, \) and \( f''(t) = 2\pi e^{-\pi t^2}(2\pi t^2 - 1). \) Prove the following:
   (a) \( M_0\{f'\} = 0. \)
   (b) \( M_0\{f''\} = M_1\{f''\} = 0. \)

4. The six vectors, \( f_1 = \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \end{bmatrix}^T, \quad f_2 = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \end{bmatrix}^T, \quad f_3 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T, \quad f_4 = \begin{bmatrix} -\cos(\pi/3) & -\sin(\pi/3) \end{bmatrix}^T, \quad f_5 = \begin{bmatrix} -\cos(\pi/3) & \sin(\pi/3) \end{bmatrix}^T, \quad f_6 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \) form a frame \( F \) for \( \mathbb{R}^2. \) Draw the frame.
   (a) Give two representations for the vector, \( x = \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \) in \( F. \)
   (b) Prove that \( x \) has an infinite number of representations in \( F. \)
   (c) Give a matrix which transforms any representation of a vector in \( F \) into its representation in the standard basis for \( \mathbb{R}^2. \)
   (d) Give a matrix which transforms a representation of any vector in the standard basis for \( \mathbb{R}^2 \) into its representation in \( F. \)

5. The continuous representation of the Haar highpass filter is
\[ h_1(t) = \frac{1}{2}[\delta(t + \Delta t) - \delta(t - \Delta t)]. \]
The continuous representation of the Haar lowpass filter is
\[ h_0(t) = \frac{1}{2}[\delta(t + \Delta t) + \delta(t - \Delta t)]. \]
Prove that
\[ H_0(s)H^*_0(s) + H_1(s)H^*_1(s) = 1 \]
where \( H_0(s) \) and \( H_1(s) \) are the Fourier transforms of \( h_0(t) \) and \( h_1(t) \).

6. The \( N+1 \) channel Haar transform matrix can be recursively defined as follows:
\[
H_N = \frac{1}{\sqrt{2}} \begin{bmatrix}
I_{N-1} & 0 \\
0 & H_{N-1}
\end{bmatrix} \begin{bmatrix}
U_N \\
L_N
\end{bmatrix}
\]
where \( U_N \) convolves a length \( 2^N \) signal with the Haar highpass filter followed by downsampling, \( L_N \) convolves a length \( 2^N \) signal with the Haar lowpass filter followed by downsampling, \( I_N \) is the identity matrix of size \( 2^N \times 2^N \) and
\[
H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix}
U_1 \\
L_1
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}.
\]
(a) Using the above definitions, derive expressions for \( H_3 \) and \( H_3^{-1} \).
(b) Compute the Haar transform of the vector \( \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{bmatrix}^T \).