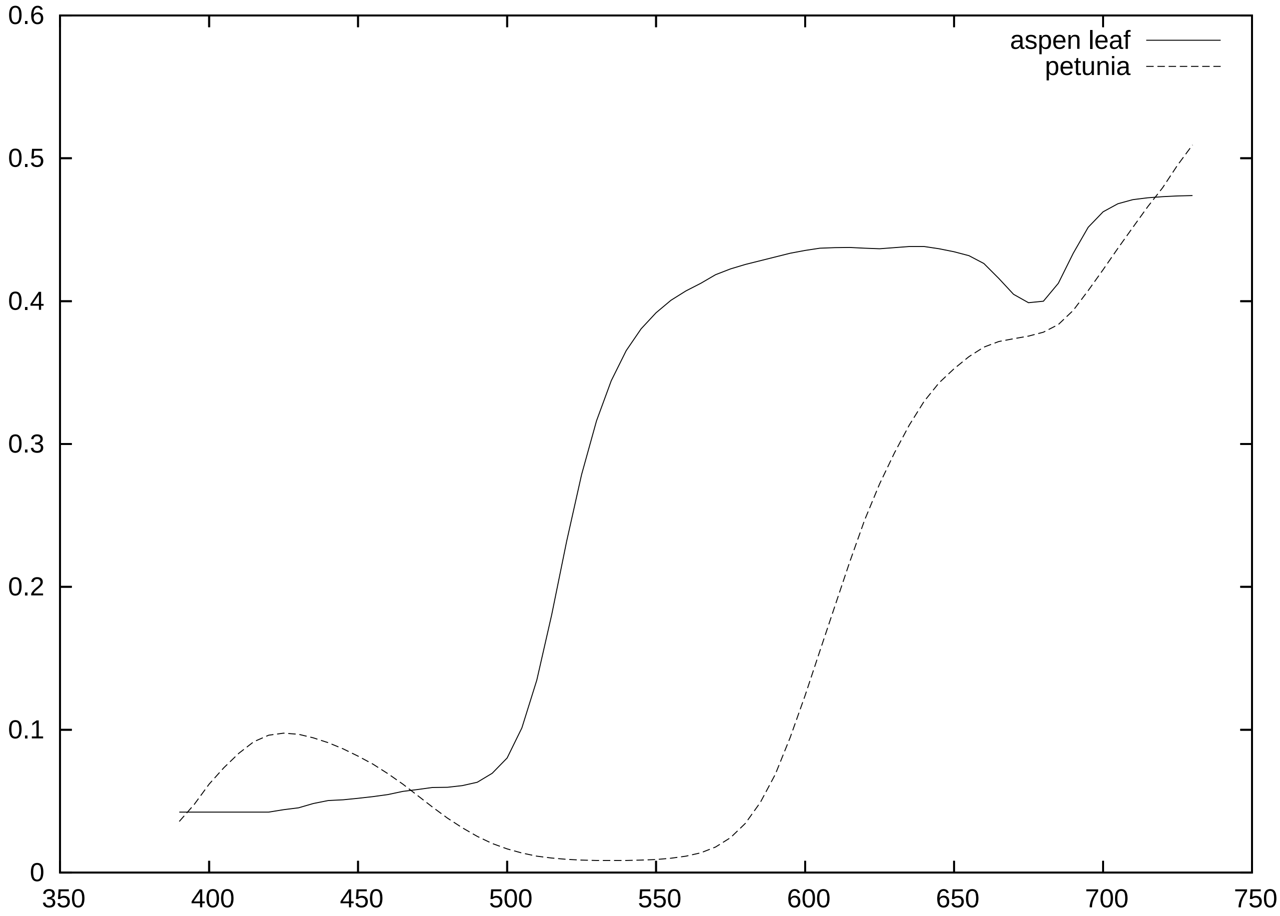

Figure 1: Reflectance function of an aspen leaf and a petunia, Clark *et al.*, 2003.

Color Vision

The L, M, S cone responses to color C are the inner products of the spectral distribution of color C and the three *cone spectral sensitivity functions*:

$$
\begin{aligned}
c_\ell(C) &= \int S_\ell(\lambda)C(\lambda)d\lambda \\
c_m(C) &= \int S_m(\lambda)C(\lambda)d\lambda \\
c_s(C) &= \int S_s(\lambda)C(\lambda)d\lambda.
\end{aligned}
$$

Metamers

Two spectral distributions, C and C', such that

$$\int S_\ell(\lambda)C(\lambda)d\lambda = \int S_\ell(\lambda)C'(\lambda)d\lambda$$
$$\int S_m(\lambda)C(\lambda)d\lambda = \int S_m(\lambda)C'(\lambda)d\lambda$$
$$\int S_s(\lambda)C(\lambda)d\lambda = \int S_s(\lambda)C'(\lambda)d\lambda$$

will be perceived to be the same color by a human observer. Such colors are termed *metamers*.

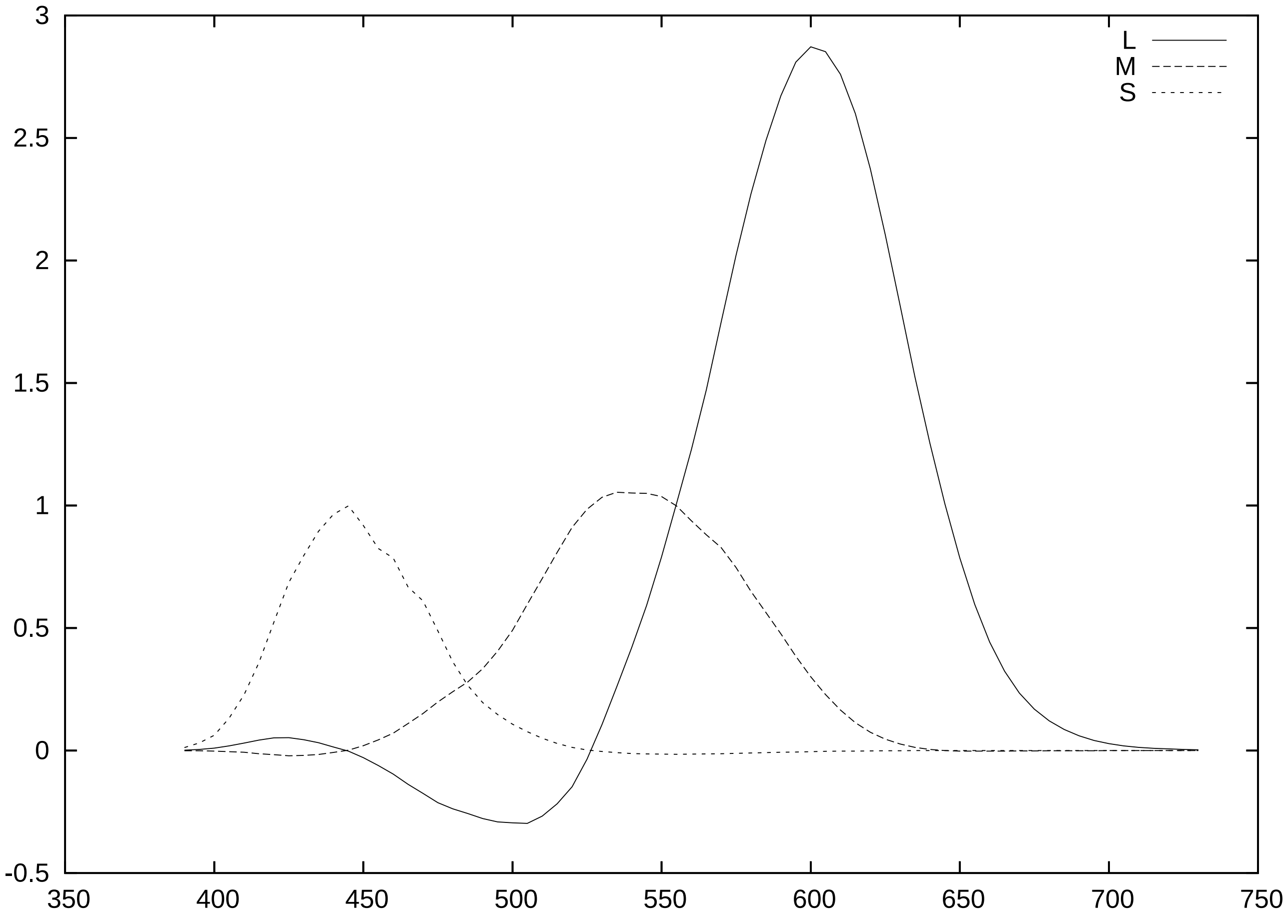

Figure 2: Cone spectral sensitivity functions $S_\ell(\lambda), S_m(\lambda)$, and $S_s(\lambda)$. Stiles and Burch, 1955.

Color Vision (contd.)

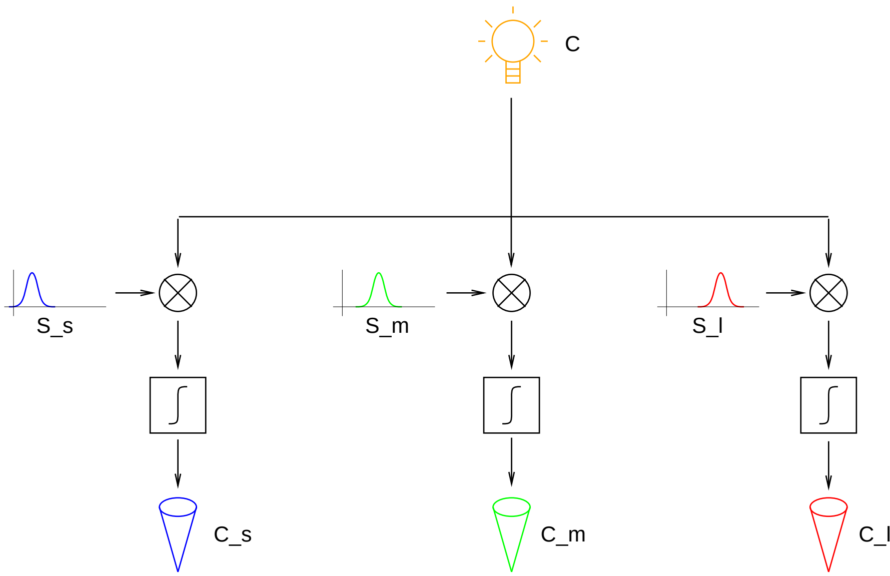

Figure 3: The L, M, S cone responses to color, C.

Linearity

If the M cone response to a light source L_1 is

$$c_m(L_1) = \int S_m(\lambda) L_1(\lambda) d\lambda$$

and to a lightsource L_2 is

$$c_m(L_2) = \int S_m(\lambda) L_2(\lambda) d\lambda$$

then (by linearity) the M cone response to a mixture of two light sources $v_1 L_1 + v_2 L_2$ is

$$c_m(v_1 L_1 + v_2 L_2) =$$

$$v_1 \int S_m(\lambda) L_1(\lambda) d\lambda + v_2 \int S_m(\lambda) L_2(\lambda) d\lambda.$$

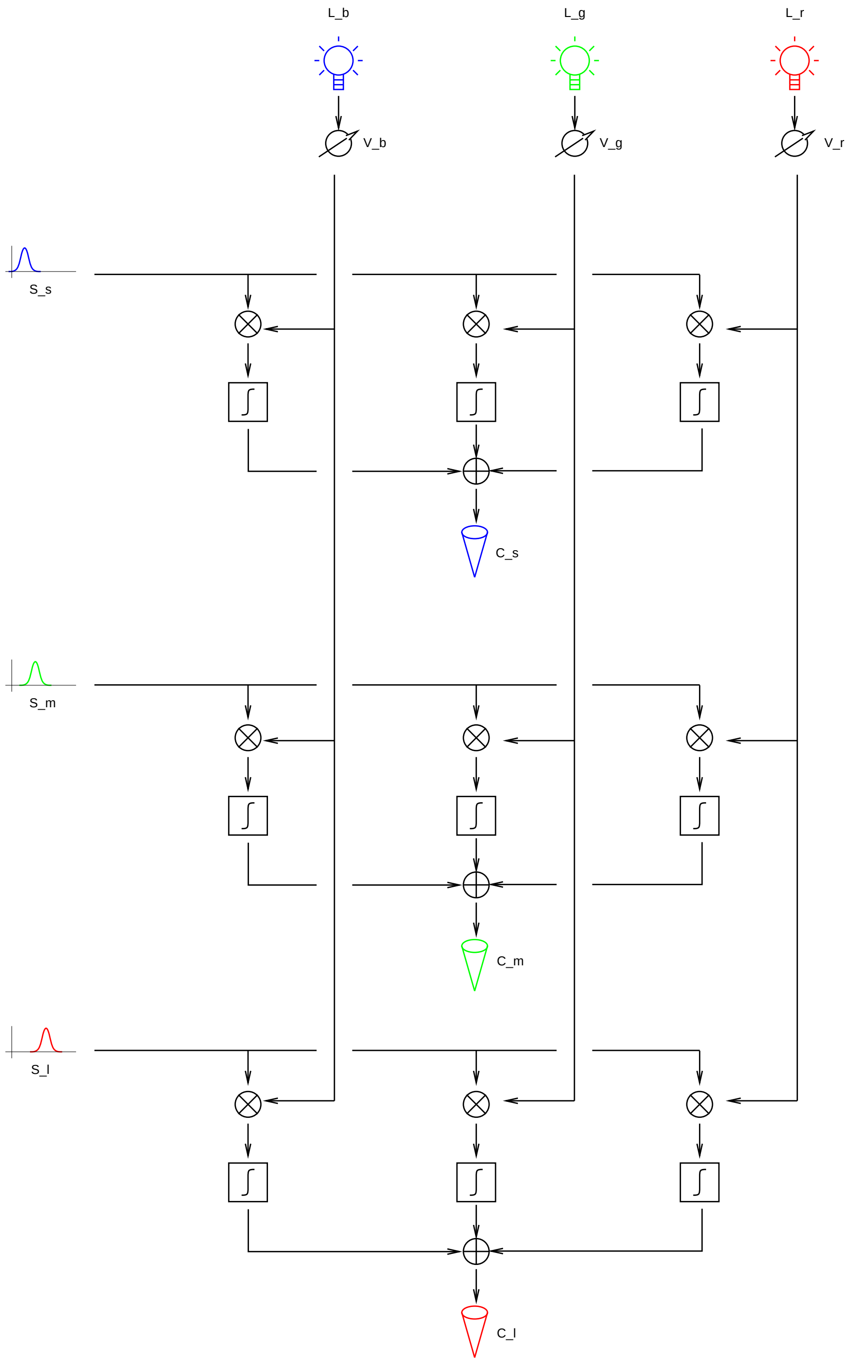

Figure 4: The L, M, S cone respones to color C can be matched by a weighted sum of three linearly independent source distributions.

Color Matching

The color, C, can be matched by a mixture of three light sources with linearly independent spectral distributions, L_r, L_g, and L_b by solving the following linear system for the appropriate color mixtures, v_r, v_g, and v_b:

$$\begin{bmatrix} c_\ell \\ c_m \\ c_s \end{bmatrix} = v_r \begin{bmatrix} m_{\ell r} \\ m_{mr} \\ m_{sr} \end{bmatrix} + v_g \begin{bmatrix} m_{\ell g} \\ m_{mg} \\ m_{sg} \end{bmatrix} + v_b \begin{bmatrix} m_{\ell b} \\ m_{mb} \\ m_{sb} \end{bmatrix}$$

where $m_{ij} = \int S_i(\lambda) L_j(\lambda) d\lambda$. Written slightly differently,

$$\begin{bmatrix} m_{\ell r} & m_{\ell g} & m_{\ell b} \\ m_{mr} & m_{mg} & m_{mb} \\ m_{sr} & m_{sg} & m_{sb} \end{bmatrix} \begin{bmatrix} v_r \\ v_g \\ v_b \end{bmatrix} = \begin{bmatrix} c_\ell \\ c_m \\ c_s \end{bmatrix}$$

we see that

$$\mathbf{v} = \mathbf{M}^{-1}\mathbf{c}.$$

Pure Sources

Things are simpler when pure sources are used:

$$
\begin{aligned}
L_r(\lambda) &= \delta(\lambda - \lambda_r) \\
L_g(\lambda) &= \delta(\lambda - \lambda_g) \\
L_b(\lambda) &= \delta(\lambda - \lambda_b).
\end{aligned}
$$

The CIE[1] standard primary sources are $\lambda_r = 700$ nm, $\lambda_g = 546.1$ nm, $\lambda_b = 435.8$ nm. The color matching equations become:

$$
\begin{bmatrix} m_{\ell r} & m_{\ell g} & m_{\ell b} \\ m_{mr} & m_{mg} & m_{mb} \\ m_{sr} & m_{sg} & m_{sb} \end{bmatrix} \begin{bmatrix} v_r \\ v_g \\ v_b \end{bmatrix} = \begin{bmatrix} c_\ell \\ c_m \\ c_s \end{bmatrix}
$$

where $m_{ij} = \int S_i(\lambda)\delta(\lambda - \lambda_j)d\lambda = S_i(\lambda_j)$.

[1] Commission Internationale de L'Eclairage.

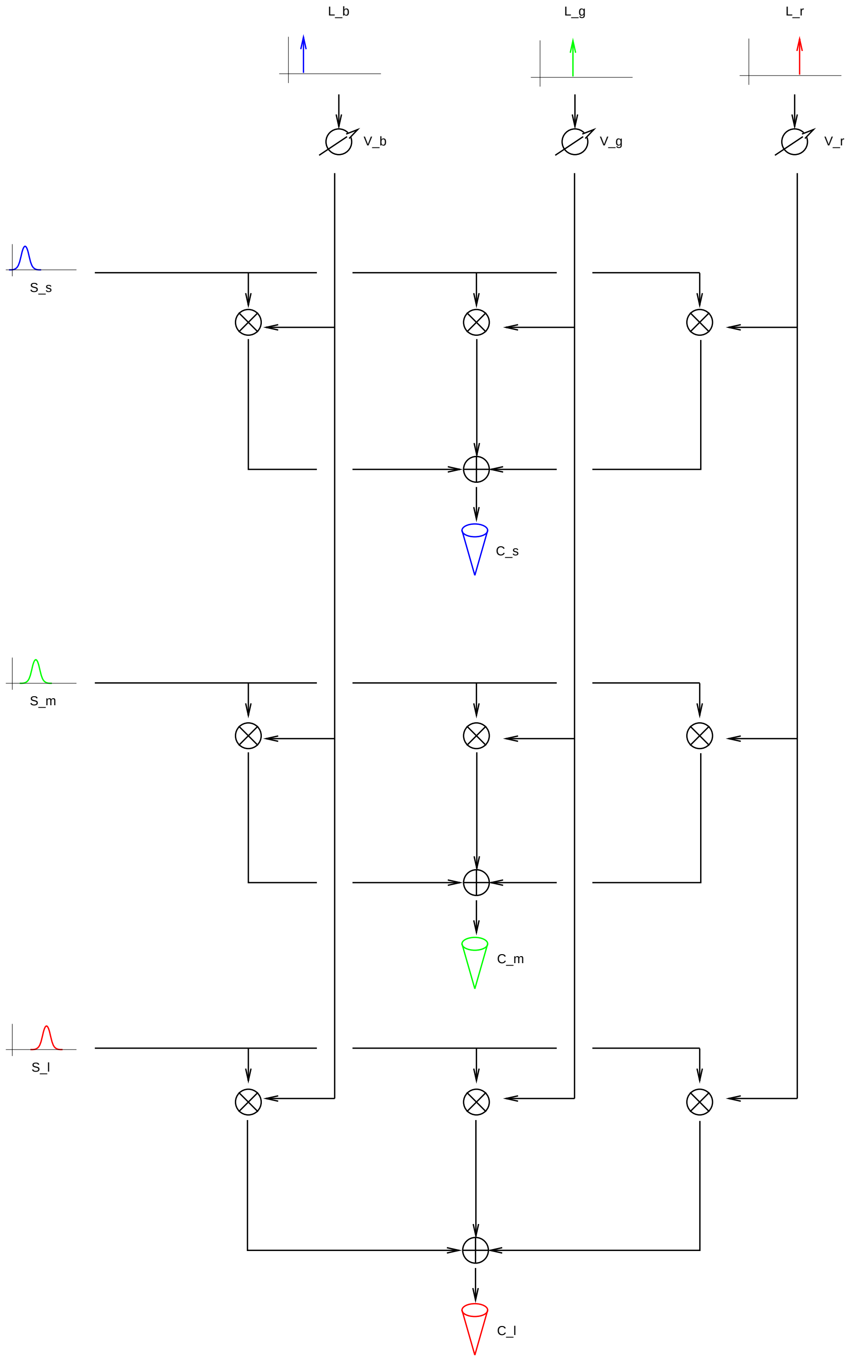

Figure 5: The L, M, S cone respones to color C can be matched by a weighted sum of three pure source distributions.

Metamers (contd.)

Consider three functions S_x, S_y, and S_z related to the cone spectral sensitivity functions S_ℓ, S_m, and S_s as follows:

$$\begin{bmatrix} S_x(\lambda) \\ S_y(\lambda) \\ S_z(\lambda) \end{bmatrix} = \begin{bmatrix} a_{\ell x} & a_{\ell y} & a_{\ell z} \\ a_{mx} & a_{my} & a_{mz} \\ a_{sx} & a_{sy} & a_{sz} \end{bmatrix} \begin{bmatrix} S_\ell(\lambda) \\ S_m(\lambda) \\ S_s(\lambda) \end{bmatrix}.$$

We will show that C and C' will be metamers with respect to S_ℓ, S_m, and S_s if they are metamers with respect to S_x, S_y, and S_z.

Metamers (contd.)

If C and C' are metamers with respect to S_x, S_y, and S_2, then

$$\int S_x(\lambda)C(\lambda)d\lambda = \int S_x(\lambda)C'(\lambda)d\lambda$$
$$\int S_y(\lambda)C(\lambda)d\lambda = \int S_y(\lambda)C'(\lambda)d\lambda$$
$$\int S_z(\lambda)C(\lambda)d\lambda = \int S_z(\lambda)C'(\lambda)d\lambda.$$

It follows that

$$\sum_{j\in\{x,y,z\}} a_{\ell j}\int S_j(\lambda)C(\lambda)d\lambda =$$
$$\sum_{j\in\{x,y,z\}} a_{\ell j}\int S_j(\lambda)C'(\lambda)d\lambda.$$

Metamers (contd.)

Rearranging things a bit we see that

$$\int \left[\sum_{j \in \{x,y,z\}} a_{\ell j} S_j(\lambda) \right] C(\lambda) d\lambda =$$

$$\int \left[\sum_{j \in \{x,y,z\}} a_{\ell j} S_j(\lambda) \right] C'(\lambda) d\lambda.$$

Now, because

$$S_\ell(\lambda) = a_{\ell x} S_x(\lambda) + a_{\ell y} S_y(\lambda) + a_{\ell z} S_z(\lambda)$$

it follows that

$$\int S_\ell(\lambda) C(\lambda) d\lambda = \int S_\ell(\lambda) C'(\lambda) d\lambda.$$

Similar arguments apply to S_m and S_s. Consequently, C and C' are metamers with respect to S_ℓ, S_m, and S_s.

Color Matching Functions

- It follows that it is possible to do color matching with any three functions related to the actual cone spectral sensitivity functions by a linear transformation.
- Suitable functions were first deduced empirically by Wright in 1929 and adopted by the CIE as a standard in 1931.

Color Matching Function Deduction

Given six pure sources at wavelengths, $\lambda_y ... \lambda_6$, the values of three color matching functions S_x, S_y, and S_z, can be deduced at these six wavelengths by repeatedly performing a simple color matching task:

- Randomly choose $v_1(C_i)$, $v_2(C_i)$, and $v_3(C_i)$.
- Adjust $v_4(C_i')$, $v_5(C_i')$, and $v_6(C_i')$ until C_i' has the same appearance as C_i.

Color Matching Function Deduction (contd.)

This yields three linear equations and eighteen unknowns:

$$\begin{bmatrix} S_x(\lambda_1) & S_x(\lambda_2) & S_x(\lambda_3) \\ S_y(\lambda_1) & S_y(\lambda_2) & S_y(\lambda_3) \\ S_z(\lambda_1) & S_z(\lambda_2) & S_z(\lambda_3) \end{bmatrix} \begin{bmatrix} v_1(C_i) \\ v_2(C_i) \\ v_3(C_i) \end{bmatrix}$$

$=$

$$\begin{bmatrix} S_x(\lambda_4) & S_x(\lambda_5) & S_x(\lambda_6) \\ S_y(\lambda_4) & S_y(\lambda_5) & S_y(\lambda_6) \\ S_z(\lambda_4) & S_z(\lambda_5) & S_z(\lambda_6) \end{bmatrix} \begin{bmatrix} v_4(C_i') \\ v_5(C_i') \\ v_6(C_i') \end{bmatrix}.$$

Repeating the color matching task six times yields a system of eighteen equations and eighteen unkowns which can be solved by Gaussian elimination.

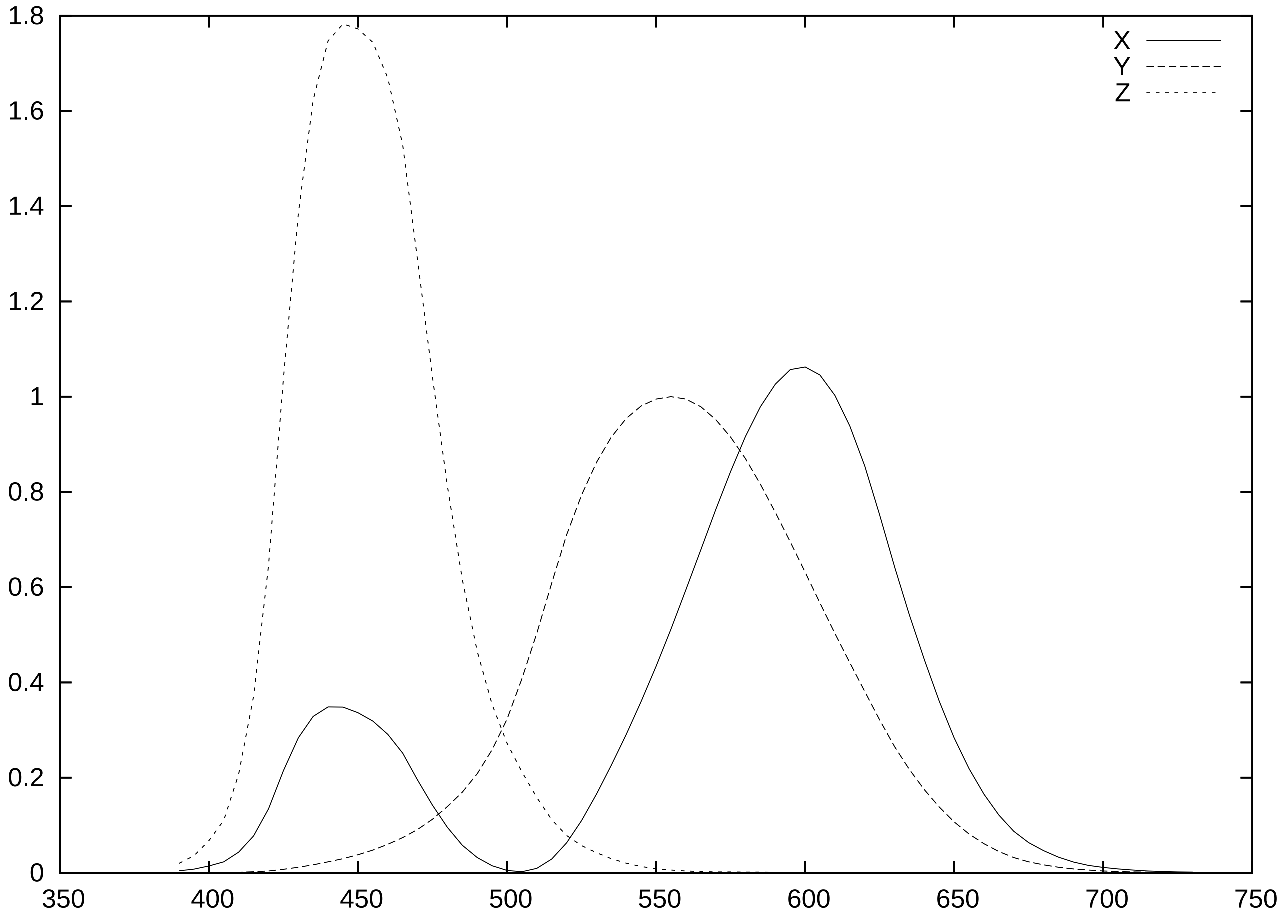

Figure 6: CIE 1931 color matching functions $S_x(\lambda), S_y(\lambda)$, and $S_z(\lambda)$. Wright, 1929.

Tristimulus Values

The *tristimulus values* for color C are the inner products of the spectral distribution of color C and the three CIE 1931 color matching functions:

$$
\begin{aligned}
X(C) &= \int S_x(\lambda)C(\lambda)d\lambda \\
Y(C) &= \int S_y(\lambda)C(\lambda)d\lambda \\
Z(C) &= \int S_z(\lambda)C(\lambda)d\lambda.
\end{aligned}
$$

The tristimulus values for white are $X(W) = Y(W) = Z(W) = 1$.

Chromaticities

The *chromaticities* for color C are computed by normalizing the tristimulus values by their sum:

$$x(C) = \frac{X(C)}{X(C)+Y(C)+Z(C)}$$
$$y(C) = \frac{Y(C)}{X(C)+Y(C)+Z(C)}$$
$$z(C) = \frac{Z(C)}{X(C)+Y(C)+Z(C)}.$$

Since the chromaticities for any color, C, always sum to one, $x(C)+y(C)+z(C)=1$, only two of the three chromaticities, *e.g.*, x and y, are independent. Plotting the set of visible colors in x and y coordinates results in the *chromaticity diagram.*

Hue, Saturation and Intensity

Another useful coordinate system is HSI, where HSI stand for *hue*, *saturation*, and *intensity*.

- *Hue* is an angular quantity which correlates closely with wavelength. For example, as hue varies between 0° and 360°, the perceived colors move through the visible spectrum, *i.e.*, red $\rightarrow$ orange $\rightarrow$ yellow $\rightarrow$ green $\rightarrow$ blue $\rightarrow$ indigo $\rightarrow$ violet $\rightarrow$ red.
- *Saturation* represents the purity of a color, *i.e.*, the absence of white. For example, red is more saturated than pink.
- *Intensity* is a measure of the overall brightness. Black is zero intensity.

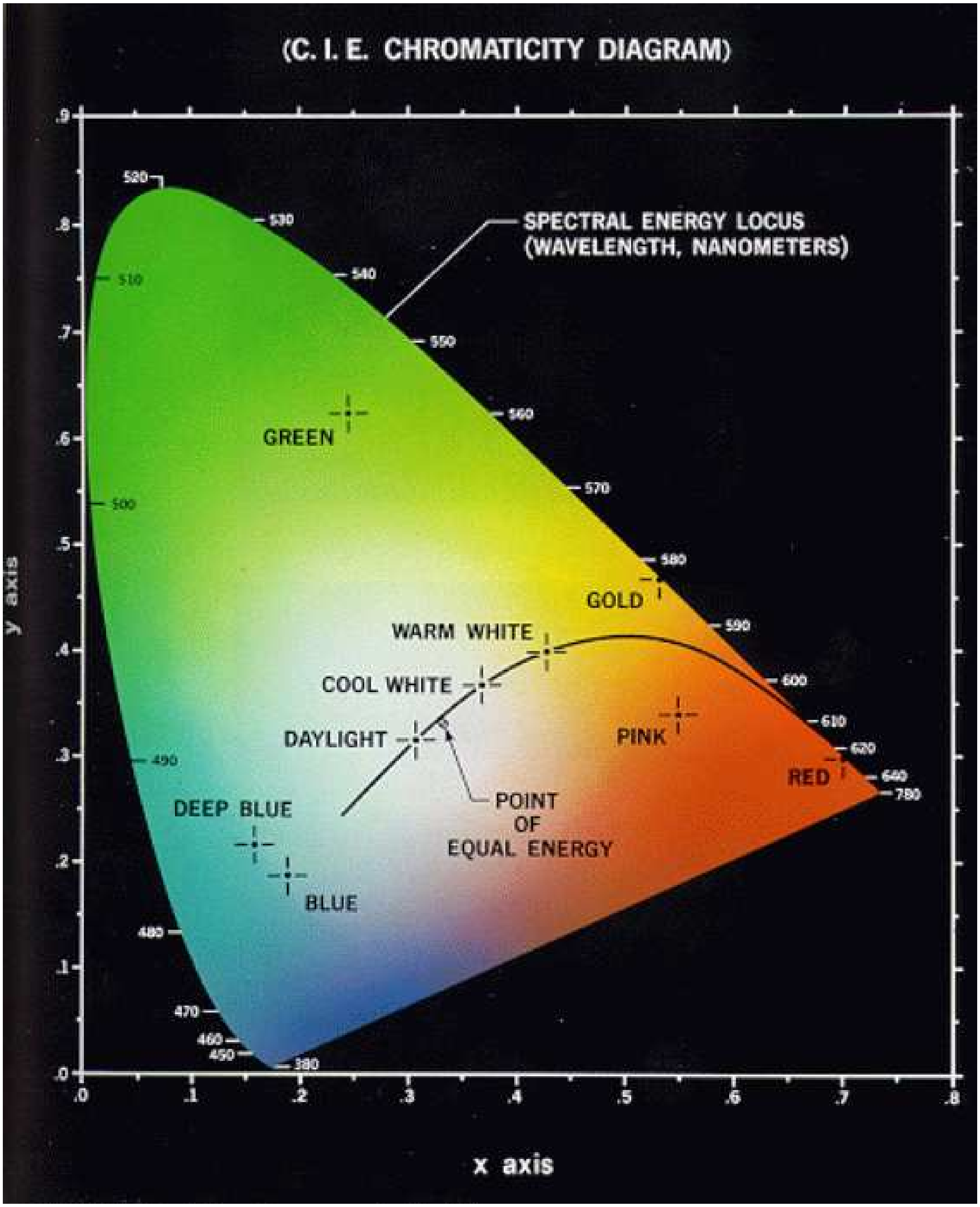

Figure 7: Chromaticity diagram. CIE, 1931.

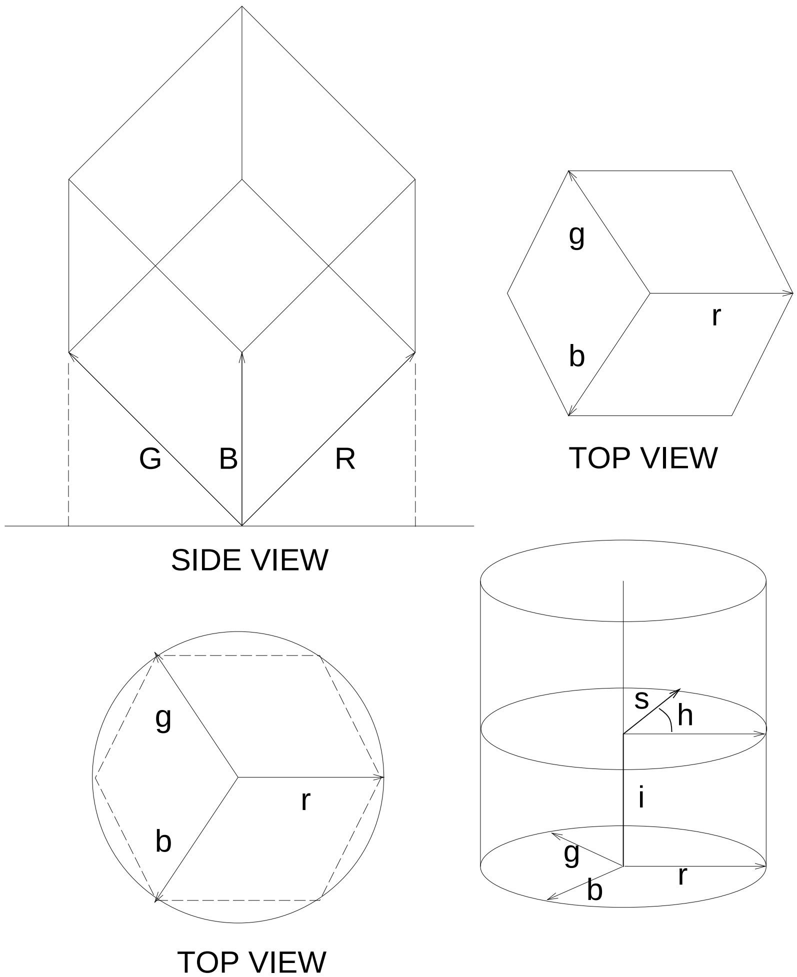

Figure 8: Converting from RGB to HSI.

Hue, Saturation and Intensity (contd.)

The following equations allow us to convert from RGB coordinates to HSI coordinates:

$$\theta = \cos^{-1}\left[\frac{\frac{1}{2}[(R-G)+(R-B)]}{\sqrt{(R-G)^2+(R-B)(G-B)}}\right]$$

$$\begin{aligned} H &= \begin{cases} \theta & \text{if } G \geq B, \\ 2\pi - \theta & \text{otherwise.} \end{cases} \\ S &= 1 - \frac{3}{R+G+B}\min(R,G,B) \\ I &= R+G+B \end{aligned}$$

Hue, Saturation and Intensity (contd.)

The following equations allow us to convert from HSI coordinates back to RGB coordinates. For $0^\circ \leq H < 120^\circ$:

$$
\begin{aligned}
R &= \frac{I}{\sqrt{3}}\left[1+\frac{S\cos(H)}{\cos(60^\circ - H)}\right] \\
G &= \sqrt{3}I - R - B \\
B &= \frac{I}{\sqrt{3}}(1-S)
\end{aligned}
$$

For $120^\circ \leq H < 240^\circ$:

$$
\begin{aligned}
R &= \frac{I}{\sqrt{3}}(1-S) \\
G &= \frac{I}{\sqrt{3}}\left[1+\frac{S\cos(H-120^\circ)}{\cos(180^\circ - H)}\right] \\
B &= \sqrt{3}I - R - G
\end{aligned}
$$

Hue, Saturation and Intensity (contd.)

For $240^\circ \leq H < 360^\circ$:

$$
\begin{aligned}
R &= \sqrt{3}I - G - B \\
G &= \frac{I}{\sqrt{3}}(1 - S) \\
B &= \frac{I}{\sqrt{3}}\left[1 + \frac{S\cos\left(H - 240^\circ\right)}{\cos(300^\circ - H)}\right]
\end{aligned}
$$