The Dirac delta function

There is a function called the *pulse*:

$$\Pi(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2} \\ 1 & \text{otherwise.} \end{cases}$$

Note that the area of the pulse is one. The *Dirac delta* function (a.k.a. the *impulse*) can be defined using the pulse as follows:

$$\delta(t) = \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \Pi\left(\frac{t}{\varepsilon}\right).$$

The impulse can be thought of as the limit of a pulse as its width goes to zero and its area is normalized to one.

Properties of the Dirac delta function

The Dirac delta function obeys the following two properties:

- *integral property*

$$\lim_{\varepsilon \longrightarrow 0} \int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1$$

- *sifting property*

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau.$$

Impulse response function

In the continuum, the output of a linear shift invariant system is given by the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Since functions remain unchanged by convolution with the impulse:

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$$

we say that the impulse is the *identity function* of linear shift invariant operators.

<u>System identification</u>

Let's say we have an unknown linear shift invariant system, *i.e.*, a black box, $\mathcal{H}$:

$$x(t) \xrightarrow{\mathcal{H}} y(t)$$

where

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- **Question** How do we find the function, h, which characterizes the linear shift invariant system?
- **Answer** Feed it an impulse and see what comes out:

$$\delta(t) \xrightarrow{\mathcal{H}} ?$$

System identification (contd.)

By commutativity and the sifting property we see that:

$$\int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau =$$
$$\int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = h(t).$$

It follows that:

$$\delta(t) \xrightarrow{\mathcal{H}} h(t).$$

For this reason, h is called the *impulse response function*. The impulse response is the first of two ways to characterize a linear shift invariant system.

Impulse Response of Shift Operator

To identify the impulse response function of the shift operator, s_Δ, we apply the shift operator to an impulse and see what comes out:

$$\delta(.) \xrightarrow{s_\Delta} \delta((.) - \Delta).$$

We conclude that $\delta((.) - \Delta)$ is the impulse response of the shift operator. It follows that to apply s_Δ to a function f, we can convolve f with $\delta((.) - \Delta)$:

$$\begin{aligned} f_\Delta(t) &= \int_{-\infty}^{\infty} f(\tau)\delta((t-\tau)-\Delta)d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\delta(t-\Delta-\tau)d\tau \\ &= f(t-\Delta). \end{aligned}$$

Impulse Response of ∇ Operator

To identify the impulse response function of the differentiation operator, ∇, we apply the differentiation operator to an impulse and see what comes out:

$$\delta(.) \xrightarrow{\nabla} \delta'(.).$$

We conclude that $\delta'(.)$, the derivative of an impulse, is the impulse response of the differentation operator. It follows that to apply ∇ to a function f, we can convolve f with $\delta'(.)$:

$$f'(t) = \int_{-\infty}^{\infty} f(\tau)\delta'(t-\tau)d\tau.$$

Harmonic signals

A harmonic signal, $\exp(j2\pi st)$, is a complex function of a real variable, t. The real part is a cosine:

$$\mathrm{Re}(e^{j2\pi st}) = \cos(2\pi st)$$

and the imaginary part is a sine:

$$\mathrm{Im}(e^{j2\pi st}) = \sin(2\pi st).$$

The transfer function

Let $x_s(t)$ and $y_s(t)$ be the input and output functions of a linear shift invariant system, $\mathcal{H}$:

$$x_s(t) \xrightarrow{\mathcal{H}} y_s(t).$$

We observe that the output function, $y_s(t)$, can be written as a product of the input function $x_s(t)$ and a function $H(s,t)$ defined as follows:

$$H(s,t) = \frac{y_s(t)}{x_s(t)}.$$

Consequently,

$$x_s(t) \xrightarrow{\mathcal{H}} H(s,t)x_s(t).$$

Since the above holds for any input function, $x_s(t)$, it holds when $x_s(t) = \exp(j2\pi st)$:

$$e^{j2\pi st} \xrightarrow{\mathcal{H}} H(s,t)e^{j2\pi st}.$$

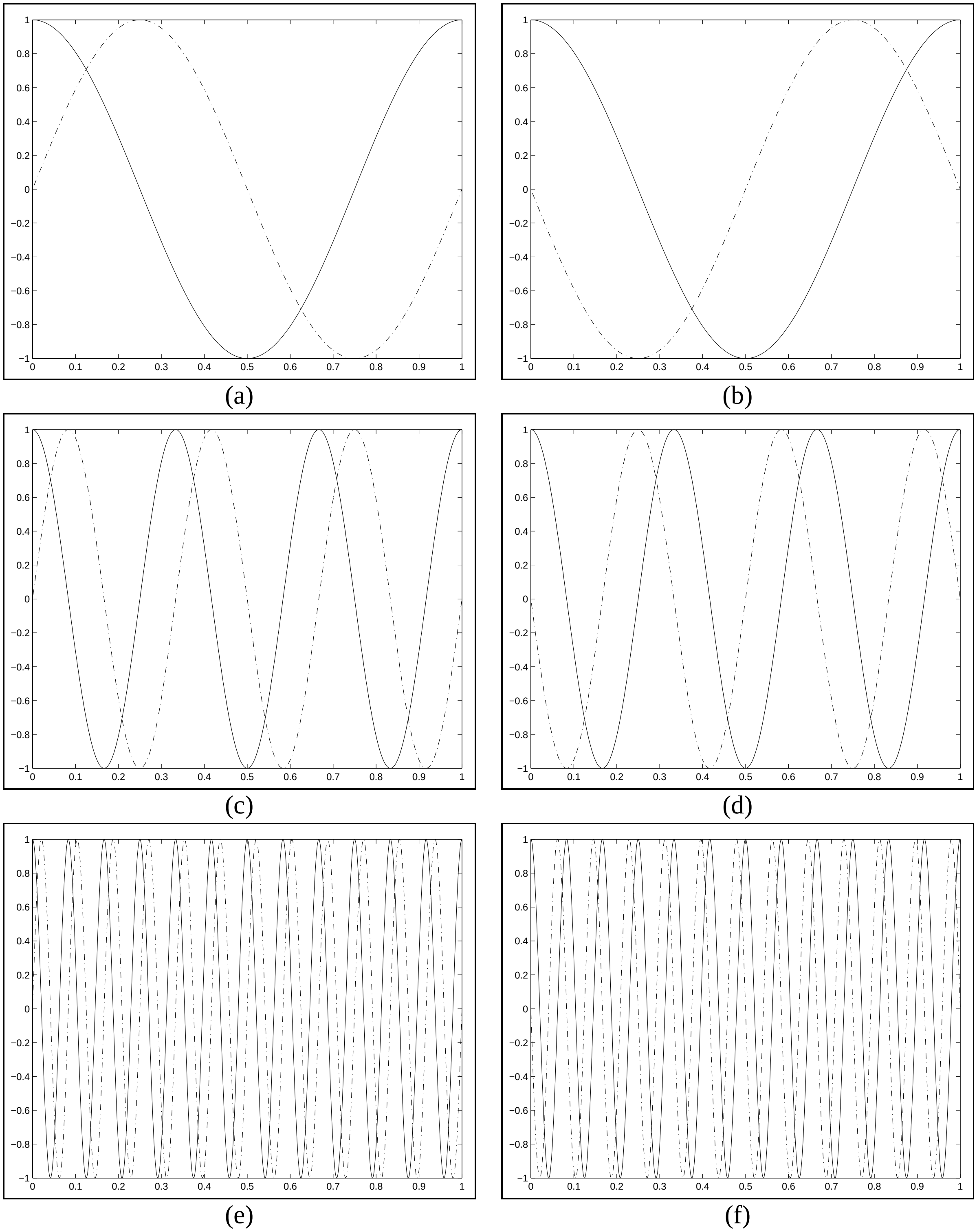

Figure 1: $\mathrm{Re}(t)$ (solid) and $\mathrm{Im}(t)$ (dashed) for harmonic signals. (a) $\exp(j2\pi t)$. (b) $\exp(-j2\pi t)$. (c) $\exp(j2\pi 3t)$. (d) $\exp(-j2\pi 3t)$. (e) $\exp(j2\pi 12t)$. (f) $\exp(-j2\pi 12t)$.

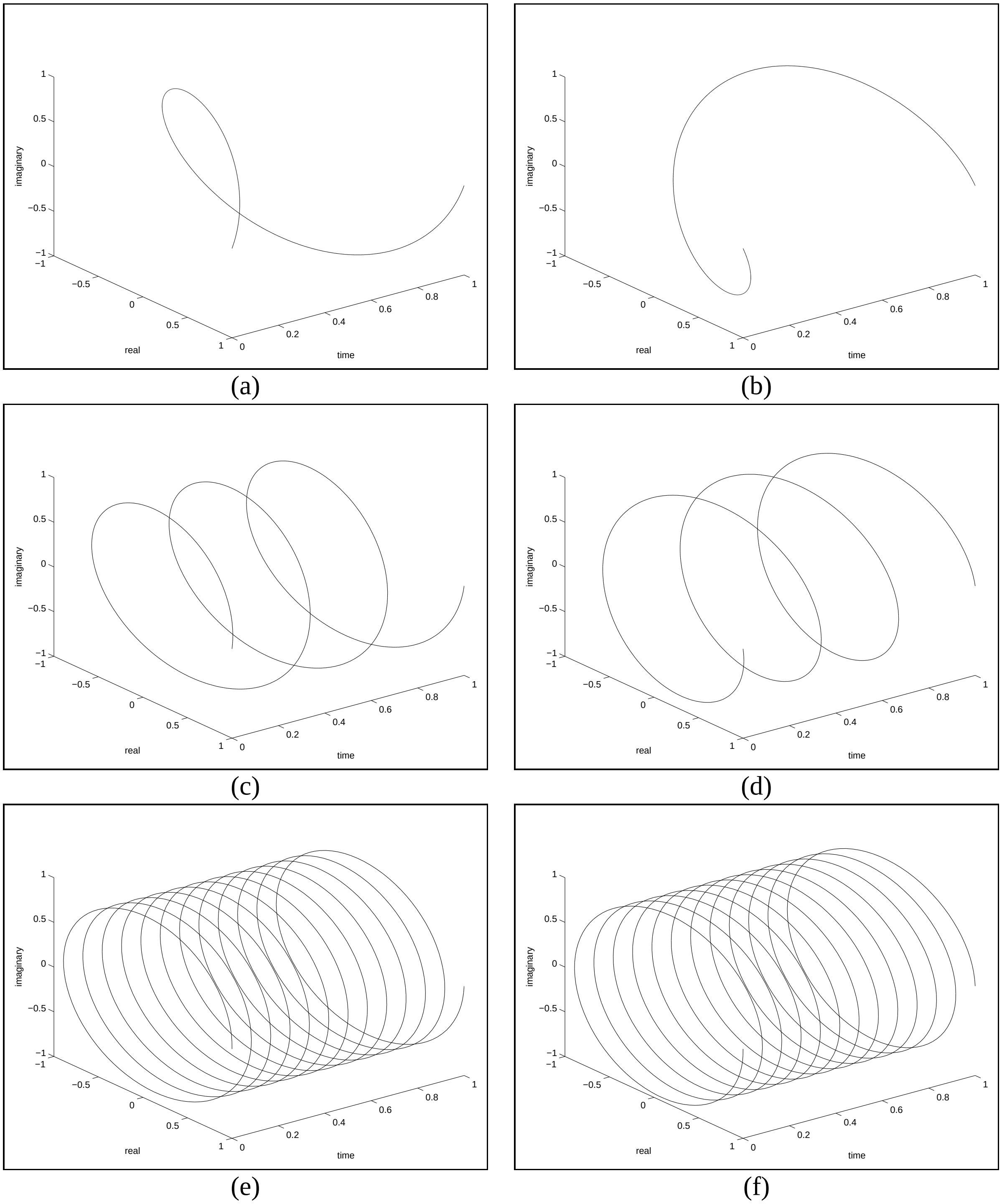

Figure 2: Harmonic signals visualized as space curve, $[\mathrm{Re}(t), \mathrm{Im}(t)]$. (a) $\exp(j2\pi t)$. (b) $\exp(-j2\pi t)$. (c) $\exp(j2\pi 3t)$. (d) $\exp(-j2\pi 3t)$. (e) $\exp(j2\pi 12t)$. (f) $\exp(-j2\pi 12t)$.

The transfer function (contd.)

Now let's shift the input:

$$e^{j2\pi s(t-\tau)} \xrightarrow{\mathcal{H}} H(s,t-\tau)e^{j2\pi s(t-\tau)}.$$

As expected, the output is shifted by the same amount. But notice that

$$\begin{aligned} e^{j2\pi s(t-\tau)} &= e^{j2\pi st}e^{-j2\pi s\tau} \\ &= e^{-j2\pi s\tau}e^{j2\pi st} \end{aligned}$$

where $e^{-j2\pi s\tau}$ is just a (complex) constant.

The transfer function (contd.)

Linearity tells us the effect of multiplying the input of a linear shift invariant system by a constant:

$$kx_s(t) \xrightarrow{\mathcal{H}} ky_s(t)$$

so that

$$e^{-j2\pi s\tau}e^{j2\pi st} \xrightarrow{\mathcal{H}} e^{-j2\pi s\tau}H(s,t)e^{j2\pi st}$$

or

$$e^{j2\pi s(t-\tau)} \xrightarrow{\mathcal{H}} H(s,t)e^{j2\pi s(t-\tau)}.$$

We can only conclude that

$$H(s,t-\tau) = H(s,t).$$

Observe that $H(s,t)$ is independent of t!

Eigenfunctions

It follows that the effect of applying a linear shift invariant operator $\mathcal{H}$ to a harmonic signal

$$e^{j2\pi st} \xrightarrow{\mathcal{H}} H(s)e^{j2\pi st}$$

is to multiply it by a complex constant $H(s)$ dependent only on frequency s. This multiplication can change the amplitude and phase of the harmonic signal, but not its frequency.

Eigenfunctions (contd.)

Written somewhat differently, the effect of a linear shift invariant operator $\mathcal{H}$ on a harmonic signal is:

$$H(s)e^{j2\pi st} = \mathcal{H}\left\{e^{j2\pi st}\right\}$$

or

$$H(s)e^{j2\pi st} = \int_{-\infty}^{\infty} e^{j2\pi s\tau} h(t-\tau)d\tau.$$

Observe the similarity between the above and the familiar equation relating eigenvector $\mathbf{x}_i$ and eigenvalue λ_i of matrix $\mathbf{A}$:

$$\lambda_i \mathbf{x}_i = \mathbf{A}\mathbf{x}_i$$

or

$$\lambda_i \left(\mathbf{x}_i\right)_j = \sum_k A_{jk} \left(\mathbf{x}_i\right)_k .$$

Because of this similarity, we say that harmonic signals are the *eigenfunctions* of linear shift invariant systems. $e^{j2\pi st}$ is like an eigenvector and $H(s)$ is like an eigenvalue.

The transfer function (contd.)

Next week, we will see that (almost) any function f can be uniquely decomposed into a weighted sum of harmonic signals, *i.e.*, eigenfunctions of $\mathcal{H}$:

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds.$$

F is called the *Fourier transform* of f.

The transfer function (contd.)

For the moment, we won't consider the problem of how to compute F. We simply observe that in the basis of eigenfunctions of $\mathcal{H}$, each component $F(s)$ of the representation of $f(t)$ is modulated by a complex constant, $H(s)$:

$$\int_{-\infty}^{\infty} H(s)F(s)e^{j2\pi st}ds = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau.$$

Like the impulse response function, h, the *transfer function*, H, completely specifies the behavior of the linear shift invariant operator $\mathcal{H}$.

The transfer function (contd.)

- **Question** What is the relationship between the impulse response function, h, and the transfer function, H?
- **Answer** H is the Fourier transform of h:

$$h(t) = \int_{-\infty}^{\infty} H(s) e^{j2\pi st} ds.$$