The Fourier Transform

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Orthonormal bases for $\mathbb{R}^n$

Let $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ and $\mathbf{v} = [v_1, v_2]^{\mathrm{T}}$ be vectors in $\mathbb{R}^2$. We define the *inner product* of $\mathbf{u}$ and $\mathbf{v}$ to be

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2.$$

We can use the inner product to define notions of *length* and *angle*. The length of $\mathbf{u}$ is given by the square root of the inner product of $\mathbf{u}$ with itself:

$$\begin{aligned} |\mathbf{u}| &= \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \\ &= \sqrt{u_1^2 + u_2^2}. \end{aligned}$$

The angle between $\mathbf{u}$ and $\mathbf{v}$ can also be defined in terms of inner product:

$$\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u}||\mathbf{v}| \cos \theta$$

where

$$\theta = \cos^{-1} \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u}||\mathbf{v}|} \right).$$

Orthogonality

An important special case occurs when

$$\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u}||\mathbf{v}| \cos\theta = 0.$$

When $\cos\theta$ equals zero, $\theta = \pi/2 = 90^\circ$.

Orthonormal bases for $\mathbb{R}^n$

Any n orthogonal vectors which are of unit length

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

form an *orthonormal basis* for $\mathbb{R}^n$. Any vector in $\mathbb{R}^n$ can be expressed as a weighted sum of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3, \ldots, \mathbf{u}_n$:

$$\mathbf{v} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + w_3\mathbf{u}_3 + \ldots + w_n\mathbf{u}_n.$$

- **Question** How do we find w_1, w_2, $w_3, \ldots, w_n$?
- **Answer** Using inner product.

Example

Consider two orthonormal bases. The first basis is defined by the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It is easy to verify that these two vectors form an orthonormal basis:

$$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = 1 \cdot 0 + 0 \cdot 1 = 0$$

$$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = 1 \cdot 1 + 0 \cdot 0 = 1$$

$$\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = 0 \cdot 0 + 1 \cdot 1 = 1.$$

Example (contd.)

The second, by the vectors $\mathbf{u}'_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\mathbf{u}'_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. It is also easy to verify that these two vectors form an orthonormal basis:

$$\left\langle \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\rangle = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0$$

$$\left\langle \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\rangle = \cos^2\theta + \sin^2\theta = 1$$

$$\left\langle \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\rangle = \cos^2\theta + \sin^2\theta = 1.$$

Example (contd.)

Let the coefficients of $\mathbf{v}$ in the first basis be w_1 and w_2:

$$\mathbf{v} = w_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

What are the coefficients of $\mathbf{v}$ in the second basis? Stated differently, what values of w_1' and w_2' satisfy:

$$\mathbf{v} = w_1' \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + w_2' \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}?$$

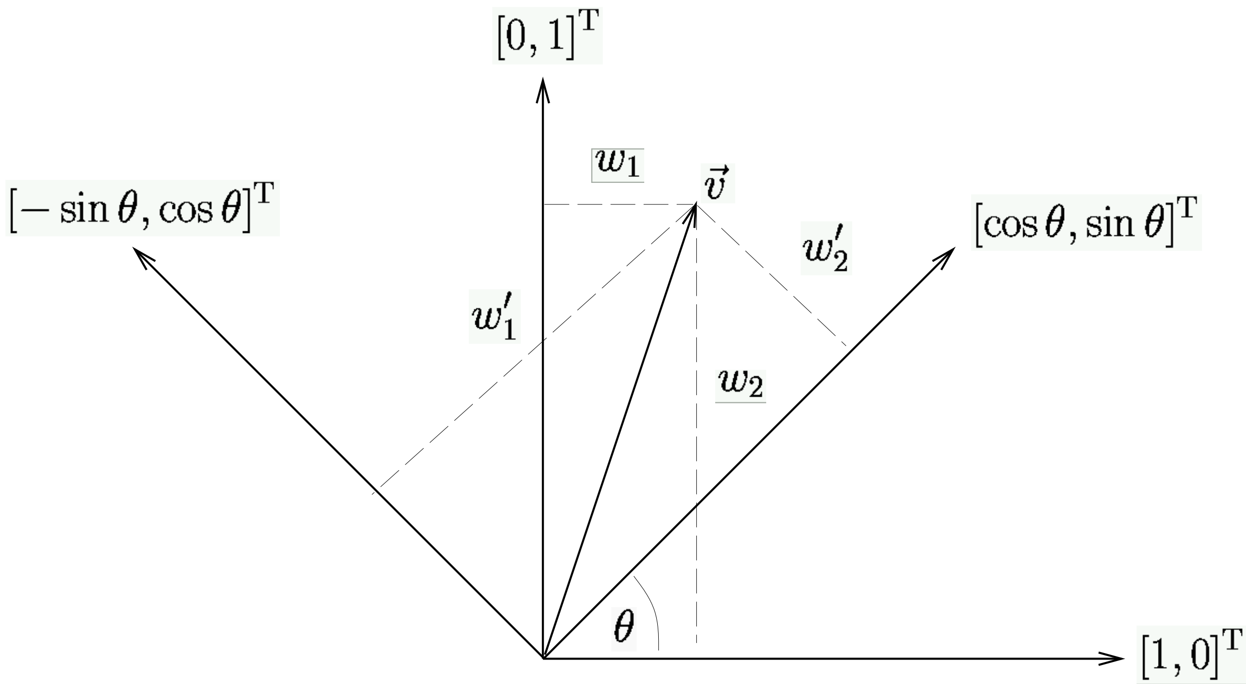

Figure 1: Change of basis.

Example (contd.)

To find w'_1 and w'_2, we use inner product:

$$w'_1 = \left\langle \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle$$

$$w'_2 = \left\langle \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle.$$

Example (contd.)

The above can be written more economically in matrix notation:

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\mathbf{w}' = \mathbf{A}\mathbf{w}.$$

If the rows of **A** are orthonormal, then **A** is an *orthonormal* matrix. Multiplying by an orthonormal matrix effects a *change of basis*. A change of basis between two orthonormal bases is a *rotation*.

Matrix transpose

If $\mathbf{A}$ rotates $\mathbf{w}$ by θ

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

then $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$ rotates $\mathbf{w}'$ by $-\theta$

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

In other words, $\mathbf{A}^{\mathrm{T}}$ undoes the action of $\mathbf{A}$, *i.e.*, they are *inverses*:

$$\begin{aligned} \mathbf{A}\mathbf{A}^{\mathrm{T}} &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

For orthonormal matrices, multiplying by the transpose undoes the *change of basis*.

Complex vectors in $\mathbb{C}^2$

$\mathbf{v} = [a_1 e^{i\theta_1}, a_2 e^{i\theta_2}]^T$ is a vector in $\mathbb{C}^2$.

- **Question** Can we define length and angle in $\mathbb{C}^2$ just like in $\mathbb{R}^2$?
- **Answer** Yes, but we need to redefine inner product:

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1^* v_1 + u_2^* v_2.$$

Note that this reduces to the inner product for $\mathbb{R}^2$ when $\mathbf{u}$ and $\mathbf{v}$ are real. The norm of a complex vector is the square root of the sum of the squares of the amplitudes. For example, for $\mathbf{v} \in \mathbb{C}^2$:

$$\begin{aligned} |\mathbf{u}| &= \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \\ &= \sqrt{u_1^* u_1 + u_2^* u_2}. \end{aligned}$$

Orthonormal bases for $\mathbb{C}^n$

- **Question** How about orthonormal bases for $\mathbb{C}^n$, do they exist?
- **Answer** Yes. If $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ when $i \neq j$ and $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 1$ when $i = j$, then the $\mathbf{u}_i$ form an orthonormal basis for $\mathbb{C}^n$.

- **Question** Do complex orthonormal matrices exist?
- **Answer** Yes, except they are called *unitary* matrices and $(\mathbf{A}^*)^{\mathrm{T}}$ undoes the action of $\mathbf{A}$. That is
$$\mathbf{A}(\mathbf{A}^*)^{\mathrm{T}} = \mathbf{I}$$
where $(\mathbf{A}^*)^{\mathrm{T}} = \mathbf{A}^{\mathrm{H}}$ is the Hermitian transpose of $\mathbf{A}$.

The space of 2π periodic functions

A function, f, is 2π periodic iff $f(t) = f(t + 2\pi)$. We can think of two complex 2π periodic functions, *e.g.*, f and g, as infinite dimensional complex vectors. Length, angle, orthogonality, and rotation (*i.e.*, change of basis) still have meaning. All that is required is that we generalize the definition of inner product:

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f^*(t)g(t)dt.$$

The length (*i.e.*, the norm) of a function is:

$$|f| = \langle f, f\rangle^{\frac{1}{2}} = \sqrt{\int_{-\pi}^{\pi} f^*(t)f(t)dt}.$$

Two functions, f and g, are orthogonal when

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f^*(t)g(t)dt = 0.$$

Scaling Property of the Impulse

The area of an impulse scales just like the area of a pulse, *i.e.*, contracting an impulse by a factor of a changes its area by a factor of $\frac{1}{|a|}$:

$$\int_{-\infty}^{\infty} \Pi(at)dt = \frac{1}{|a|} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \delta(at)dt.$$

It follows that:

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} |a|\delta(at)dt = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1.$$

Since the impulse is defined by the above integral property, we conclude that:

$$|a|\delta(at) = \delta(t).$$

Shah basis

The *Shah* function is a train of impulses:

$$\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n).$$

We can use the scaling property of the impulse to define a 2π periodic Shah function:

$$\begin{aligned}
\frac{1}{2\pi}\text{III}\left(\frac{t}{2\pi}\right) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{2\pi}-n\right) \\
&= \frac{2\pi}{2\pi}\sum_{n=-\infty}^{\infty} \delta\left(2\pi\left(\frac{t}{2\pi}-n\right)\right) \\
&= \sum_{n=-\infty}^{\infty} \delta\left(t-2\pi n\right).
\end{aligned}$$

Shah basis (contd.)

Consider the infinite set of 2π periodic Shah functions, $\frac{1}{2\pi}\text{III}\left(\frac{t-\tau}{2\pi}\right)$, for $-\pi \leq \tau < \pi$. Because $\frac{1}{2\pi}\text{III}\left(\frac{t-\tau}{2\pi}\right) = \delta(t-\tau)$ for $-\pi \leq t \leq \pi$ it follows that

$$\left\langle \frac{1}{2\pi}\text{III}\left(\frac{t-\tau_1}{2\pi}\right), \frac{1}{2\pi}\text{III}\left(\frac{t-\tau_2}{2\pi}\right) \right\rangle$$

$$= \int_{-\pi}^{\pi} \delta(t-\tau_1)\,\delta(t-\tau_2)\,dt$$

equals 0 when $\tau_1 \neq \tau_2$ and equals $\int_{-\pi}^{\pi} \delta(t-\tau_1)dt = 1$ when $\tau_1 = \tau_2$. It follows that the infinite set of 2π periodic Shah functions, $\frac{1}{2\pi}\text{III}\left(\frac{t-\tau}{2\pi}\right)$, for $-\pi \leq \tau < \pi$ form an orthonormal basis for the space of 2π periodic functions.

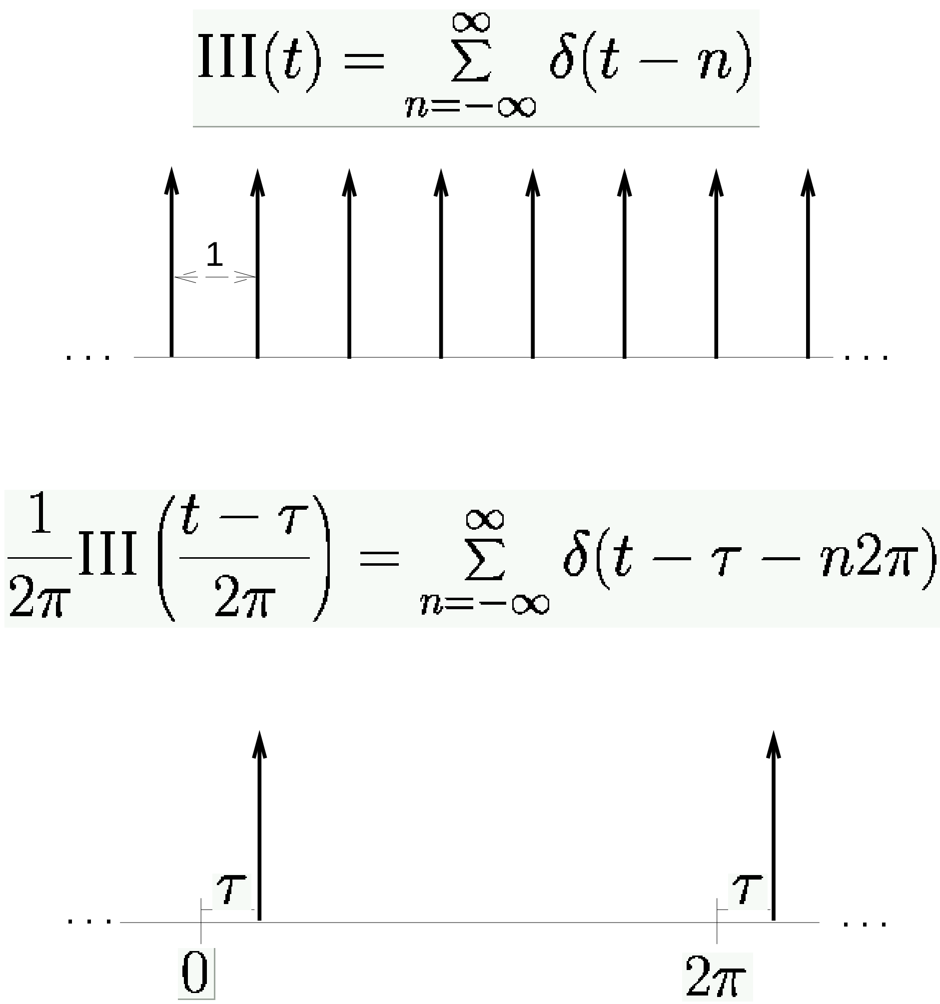

Figure 2: Making a 2π periodic Shah function.

Shah basis (contd.)

- **Question** How do we find the coefficients, $w(\tau)$, representing $f(t)$ in the Shah basis? How do we find $w(\tau)$ such that

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(\tau)\mathrm{III}\left(\frac{t-\tau}{2\pi}\right)d\tau?$$

- **Answer** Take inner products of f with the infinite set of 2π periodic Shah functions:

$$w(\tau) = \left\langle \frac{1}{2\pi}\mathrm{III}\left(\frac{t-\tau}{2\pi}\right), f(t)\right\rangle.$$

Shah basis (contd.)

Because $\frac{1}{2\pi}\text{III}\left(\frac{t-\tau}{2\pi}\right) = \delta(t-\tau)$ for $-\pi \leq t \leq \pi$ it follows that

$$\begin{aligned} w(\tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\text{III}\left(\frac{t-\tau}{2\pi}\right) dt \\ &= \int_{-\pi}^{\pi} f(t)\delta(t-\tau)\, dt \end{aligned}$$

which by the sifting property of the impulse is just:

$$w(\tau) = f(\tau).$$

We see that the coefficients of f in the Shah basis are just f itself!

Harmonic signal basis

- **Question** How long is a harmonic signal?
- **Answer** The length of a harmonic signal is

$$
\begin{aligned}
|e^{j\omega t}| &= \langle e^{j\omega t}, e^{j\omega t} \rangle^{\frac{1}{2}} \\
&= \left(\int_{-\pi}^{\pi} e^{-j\omega t} e^{j\omega t} dt \right)^{\frac{1}{2}} \\
&= \left(\int_{-\pi}^{\pi} dt \right)^{\frac{1}{2}} \\
&= \sqrt{2\pi}.
\end{aligned}
$$

Harmonic signal basis (contd.)

- **Question** What is the angle between two harmonic signals with integer frequencies?
- **Answer** The angle between two harmonic signals with integer frequencies is

$$\begin{aligned}\langle e^{j\omega_1 t}, e^{j\omega_2 t}\rangle &= \int_{-\pi}^{\pi} e^{-j\omega_1 t} e^{j\omega_2 t} dt \\ &= \left[\frac{e^{j(\omega_2-\omega_1)t}}{j(\omega_2-\omega_1)}\right]\Bigg|_{-\pi}^{\pi}.\end{aligned}$$

Since this function is the same at $-\pi$ and π (for all integers ω_1 and ω_2), we conclude that

$$\langle e^{j\omega_1 t}, e^{j\omega_2 t}\rangle = 0$$

when ω_1 and ω_2 are integers and $\omega_1 \neq \omega_2$.

Fourier Series of 2π Periodic Functions

It follows that the infinite set of harmonic signals, $\frac{1}{\sqrt{2\pi}}e^{j\omega t}$ for integer ω and $-\infty \leq \omega \leq \infty$ form an orthonormal basis for the space of 2π periodic functions.

- **Question** What are the coefficients of f in the harmonic signal basis?
- **Answer** Take inner products of f with the infinite set of harmonic signals.

This is the *analysis formula* for Fourier series:

$$\begin{aligned} F(\omega) &= \left\langle \frac{1}{\sqrt{2\pi}}e^{j\omega t}, f \right\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-j\omega t} dt \end{aligned}$$

for integer frequency, ω.

Fourier Series of 2π Periodic Functions (contd.)

The function can be reconstructed using the *synthesis formula* for Fourier series:

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} F(\omega) e^{j\omega t}.$$

Fourier Series Example

The Fourier series for the Shah basis function

$$f(t) = \frac{1}{2\pi}\text{III}\left(\frac{t}{2\pi}\right)$$

is

$$\begin{aligned} F(\omega) &= \left\langle \frac{1}{\sqrt{2\pi}} e^{j\omega t}, \frac{1}{2\pi}\text{III}\left(\frac{t}{2\pi}\right) \right\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \delta(t) e^{j\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

Consequently

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \sum_{\omega=-\infty}^{\infty} F(\omega) e^{j\omega t} \\ &= \frac{1}{2\pi} \sum_{\omega=-\infty}^{\infty} e^{j\omega t}. \end{aligned}$$

Deep Thought

The analysis formula for Fourier series effects a change of basis. It is a **rotation** in the space of 2π periodic functions. The synthesis formula undoes the change of basis. It is the opposite rotation.

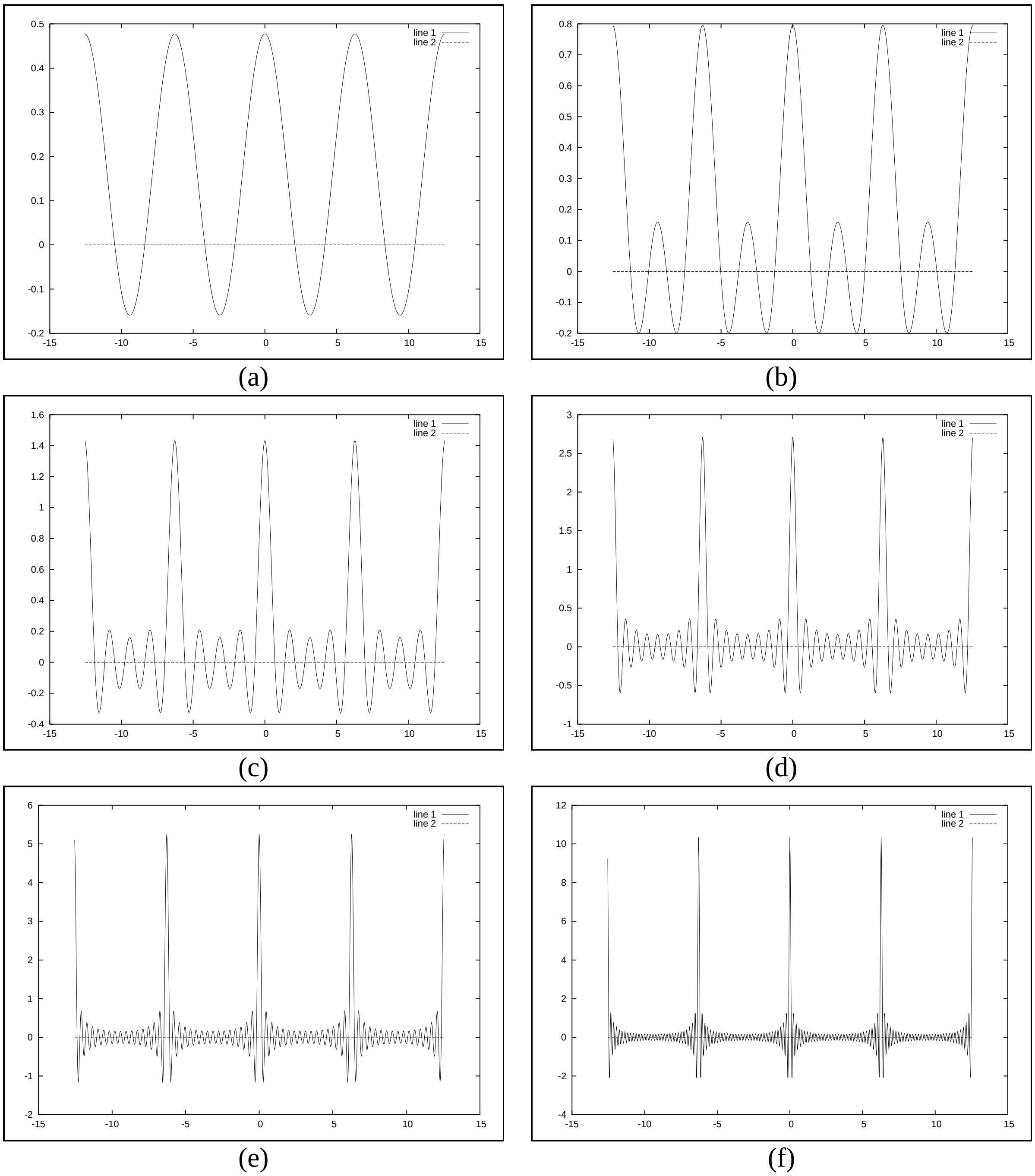

Figure 3: $\mathrm{Re}(t)$ (solid) and $\mathrm{Im}(t)$ (dashed) of truncated Fourier series for Shah basis function. (a) $-1 \leq \omega \leq 1$ (b) $-2 \leq \omega \leq 2$ (c) $-4 \leq \omega \leq 4$ (d) $-8 \leq \omega \leq 8$ (e) $-16 \leq \omega \leq 16$ (f) $-32 \leq \omega \leq 32$.

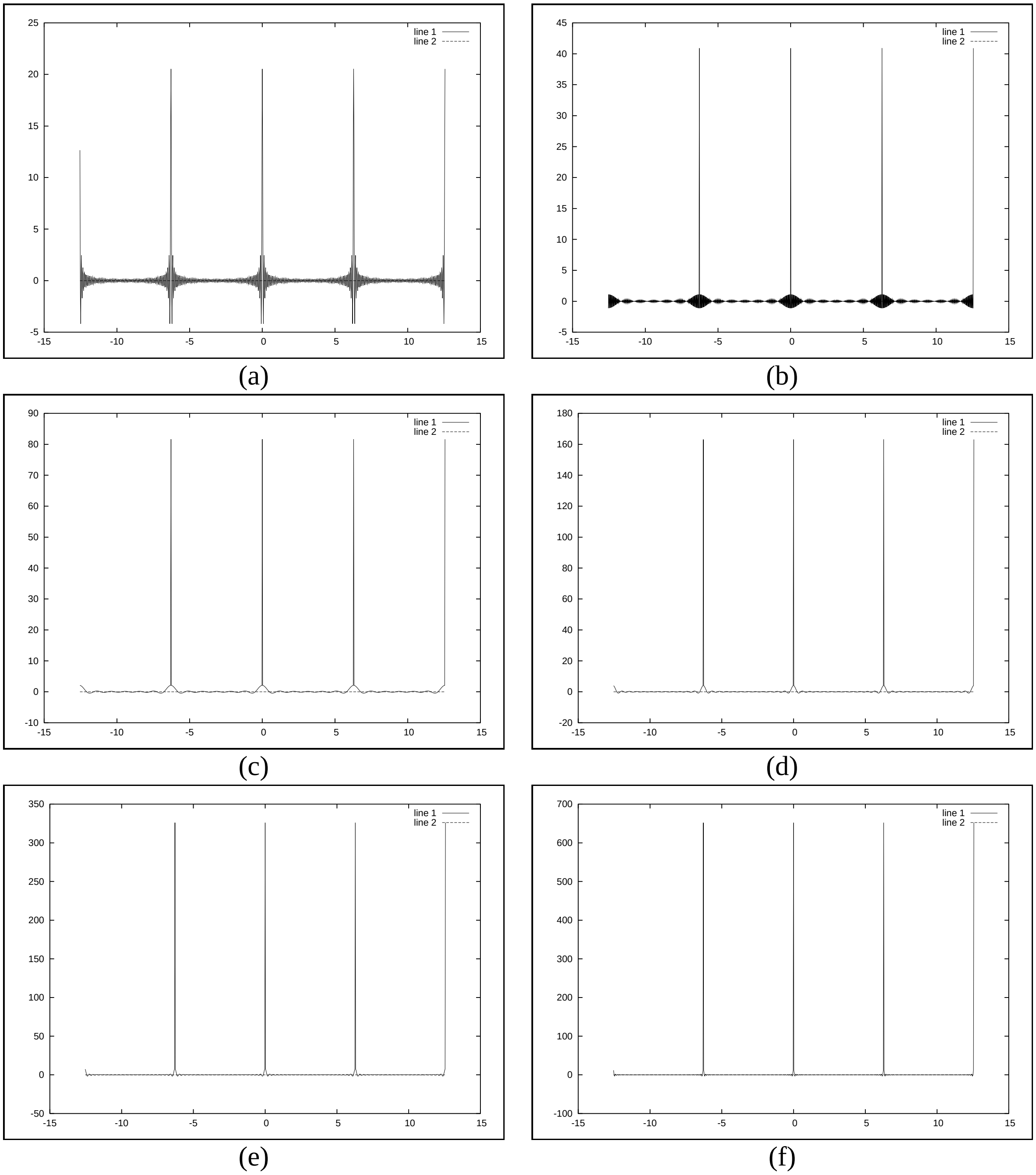

Figure 4: Re(t) (solid) and Im(t) (dashed) of truncated Fourier series for Shah basis function. (a) $-64 \leq \omega \leq 64$ (b) $-128 \leq \omega \leq 128$ (c) $-256 \leq \omega \leq 256$ (d) $-512 \leq \omega \leq 512$ (e) $-1024 \leq \omega \leq 1024$ (f) $-2048 \leq \omega \leq 2048$.

Fourier Series of T-Periodic Functions

A function, f, is T-periodic iff $f(t) = f(t+T)$.

- **Analysis formula**

$$
\begin{aligned}
F(\omega) &= \left\langle \frac{\sqrt{2\pi}}{T} e^{j2\pi\omega t/T}, f \right\rangle \\
&= \frac{\sqrt{2\pi}}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi\omega t/T} dt
\end{aligned}
$$

  for integer frequency, ω.

- **Synthesis formula**

$$
f(t) = \frac{\sqrt{2\pi}}{T} \sum_{\omega=-\infty}^{\infty} F(\omega) e^{j2\pi\omega t/T}
$$

Observe that if we substitute $T = 2\pi$ in the above expressions, we get the formulas for 2π periodic functions.

The Fourier Transform

Functions with finite length are termed *square integrable*.

$$
\begin{aligned}
|f| &= \sqrt{\int_{-\infty}^{\infty} |f(t)|^2 dt} \\
&= \sqrt{\int_{-\infty}^{\infty} f^*(t) f(t) dt} \\
&< \infty.
\end{aligned}
$$

For square integrable functions, we can take the limit of the Fourier series for T-periodic functions as $T \longrightarrow \infty$, in which case, it is possible to show that...

The Fourier Transform (contd.)

- **Analysis formula**

$$\begin{aligned} F(s) &= \langle e^{j2\pi st}, f \rangle \\ &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \end{aligned}$$

- **Synthesis formula**

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds$$