DCT Basis Functions

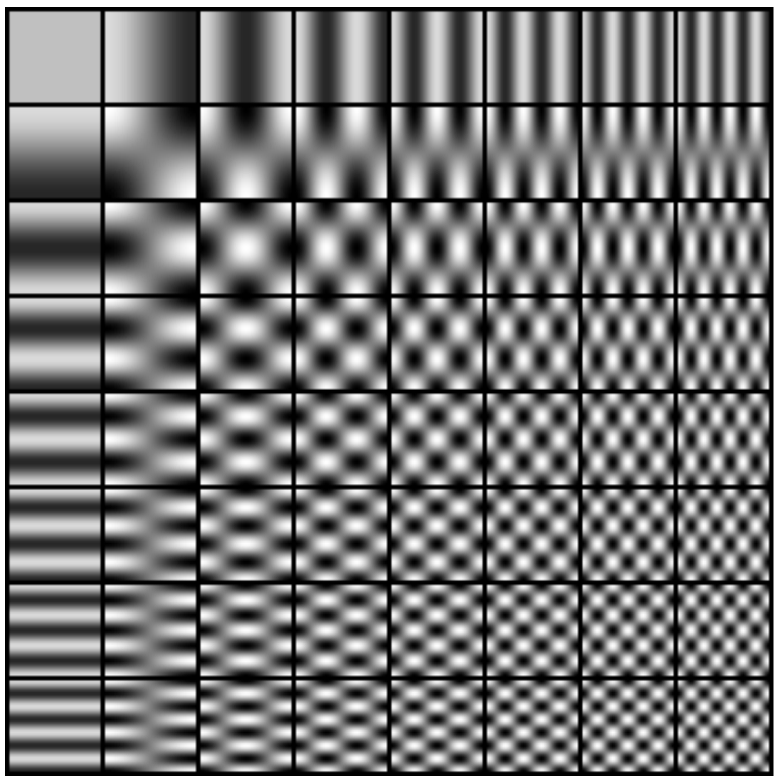

Figure 1: Basis functions of Discrete Cosine Transform (DCT)

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Simple Cell Receptive Fields

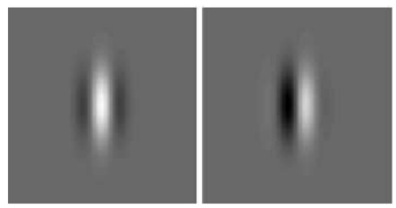

Figure 2: Cosine (left) and sine gratings (right) in Gaussian envelopes, known as *Gabor functions*, closely resemble the receptive fields of simple cells in primary visual cortex (V1). Gabor functions at a range of scales and orientations are centered at all positions (x, y) in the visual field.

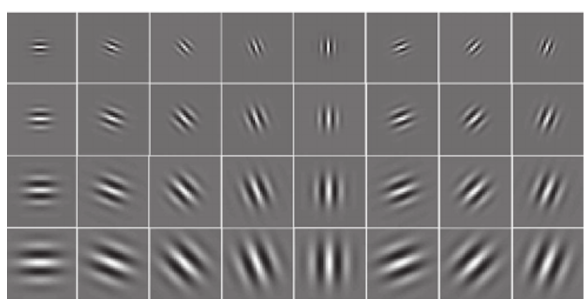

Figure 3: Cosine Gabor functions of different scales, $\log r$, and orientations, θ.

Frames vs. Bases

- A set of vectors form a *basis* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$ and are linearly independent.
- A set of $N \geq M$ vectors form a *frame* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$.

Advantages of Frame Representations

- Using bases $\mathcal{B}$ it possible to build sparse, invertible representations.
- Using frames $\mathcal{F}$ it is possible to build sparse, invertible representations that are also *Euclidean equivariant*.

Euclidean Equivariance

Primary visual cortex uses a frame operator $\mathcal{F}$ to transform an input representation, $I : \mathbb{R}^2 \rightarrow \mathbb{R}$, into an output representation of higher dimensionality, $O : \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{S}^1 \rightarrow \mathbb{R}$:

$$I\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \stackrel{\mathcal{F}}{\rightarrow} O\left(\begin{bmatrix} x \\ y \\ \log r \\ \theta \end{bmatrix}\right).$$

A *Euclidean transformation*, $\mathcal{T}$, takes an input representation and returns the same representation rotated, translated and scaled:

$$I\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \stackrel{\mathcal{T}}{\rightarrow} I\left(\begin{bmatrix} s\,(x\cos\phi + y\sin\phi) + u \\ s\,(y\sin\phi - x\cos\phi) + v \end{bmatrix}\right).$$

Euclidean Equivariance (contd.)

An operator, $\mathcal{F}$, is Euclidean equivariant, iff it commutes with $\mathcal{T}$. This property can be depicted using a commutative diagram:

$$\begin{array}{ccc} I & \xrightarrow{\mathcal{T}} & \mathcal{T}I \\ \downarrow\mathcal{F} & & \downarrow\mathcal{F} \\ \mathcal{F}I & \xrightarrow{\mathcal{T}'} & O \end{array}$$

where $\mathcal{T}'$ is the corresponding transformation of the *output representation* of higher dimensionality:

$$O\left(\begin{bmatrix} x \\ y \\ \log r \\ \theta \end{bmatrix}\right) \stackrel{\mathcal{T}'}{\to} O\left(\begin{bmatrix} s\,(x\cos\phi + y\sin\phi) + u \\ s\,(y\sin\phi - x\cos\phi) + v \\ \log r + \log s \\ \theta + \phi \end{bmatrix}\right).$$

Synthesis Matrix

Let $\mathcal{B}$ consist of the M basis vectors, $\mathbf{b}_1 \ldots \mathbf{b}_M \in \mathbb{R}^M$. Let $\{\mathbf{y}\}_{\mathcal{B}} \in \mathbb{R}^M$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\mathcal{B}$. It follows that

$$\mathbf{y} = \mathbf{B}\{\mathbf{y}\}_{\mathcal{B}}$$

where the *synthesis matrix*, $\mathbf{B}$, is the $M \times M$ matrix,

$$\mathbf{B} = \left[\, \mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_M \,\right].$$

where $\mathbf{b}_i$ is column i of $\mathbf{B}$.

Analysis Matrix

To find the representation of the vector $\mathbf{y}$ in the basis $\mathcal{B}$ we multiply $\mathbf{y}$ by the *analysis matrix* $\mathbf{B}^{-1}$:

$$\{\mathbf{y}\}_{\mathcal{B}} = \mathbf{B}^{-1}\mathbf{y}.$$

The components of the representation of $\mathbf{y}$ in $\mathcal{B}$ are inner products of $\mathbf{y}$ with the rows of $\mathbf{B}^{-1}$:

$$\mathbf{B}^{-1} = \begin{bmatrix} \widetilde{\mathbf{b}}_1^{\mathrm{T}} \\ \hline \widetilde{\mathbf{b}}_2^{\mathrm{T}} \\ \hline \vdots \\ \widetilde{\mathbf{b}}_M^{\mathrm{T}} \end{bmatrix}.$$

where $\widetilde{\mathbf{b}}_i^{\mathrm{T}}$ is *row* i of $\mathbf{B}^{-1}$.

Dual Basis

The transposes of these row vectors form a *dual basis*, $\widetilde{\mathcal{B}}$, with synthesis matrix:

$$(\mathbf{B}^{-1})^{\mathrm{T}} = \left[\, \widetilde{\mathbf{b}}_1 \mid \widetilde{\mathbf{b}}_2 \mid \ldots \mid \widetilde{\mathbf{b}}_M \,\right]$$

and analysis matrix:

$$\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} \mathbf{b}_1^{\mathrm{T}} \\ \hline \mathbf{b}_2^{\mathrm{T}} \\ \hline \vdots \\ \mathbf{b}_M^{\mathrm{T}} \end{bmatrix}.$$

The relationship between the vectors of the primal ($\mathcal{B}$) and dual ($\widetilde{\mathcal{B}}$) bases is:

$$\left\langle \mathbf{b}_i \, , \, \widetilde{\mathbf{b}}_j \right\rangle = \delta_{ij}.$$

The *biorthogonality* of the columns of $\mathbf{B}$ and the rows of $\mathbf{B}^{-1}$ follows immediately from the definition of matrix inverse.

Example

Recall that any $N \times N$ matrix, $\mathbf{P}$, with N distinct *eigenvalues*, λ_i, can be factored into a product of three matrices:

$$\mathbf{P} = \mathbf{X}\Lambda\mathbf{Y}^{\mathrm{T}}$$

where the columns of

$$\mathbf{X} = \left[\ \mathbf{x}_1 \mid \mathbf{x}_2 \mid \ \ldots \ \mid \mathbf{x}_M\ \right]$$

are *right eigenvectors* satisfying $\lambda_i \mathbf{x}_i = \mathbf{P}\,\mathbf{x}_i$ and the rows of

$$\mathbf{Y}^{\mathrm{T}} = \begin{bmatrix} \mathbf{y}_1^{\mathrm{T}} \\ \hline \mathbf{y}_2^{\mathrm{T}} \\ \hline \vdots \\ \mathbf{y}_M^{\mathrm{T}} \end{bmatrix}$$

are *left eigenvectors* satisfying $\lambda_i \mathbf{y}_i^{\mathrm{T}} = \mathbf{y}_i^{\mathrm{T}}\mathbf{P}$ and Λ is a diagonal matrix of eigenvalues where $\Lambda_{ii} = \lambda_i$.

Example (contd.)

Because $\mathbf{X}$ and $\mathbf{Y}^{\mathrm{T}}$ are inverses:

$$\langle \mathbf{x}_i \,,\, \mathbf{y}_j \rangle = \delta_{ij}.$$

Consequently, the right and left eigenvectors form primal basis $\mathcal{X}$ and dual basis $\mathcal{Y}$. We take inner products with the left eigenvectors $\mathcal{Y}$ to find the representation in the basis of right eigenvectors $\mathcal{X}$ and *vice versa*.

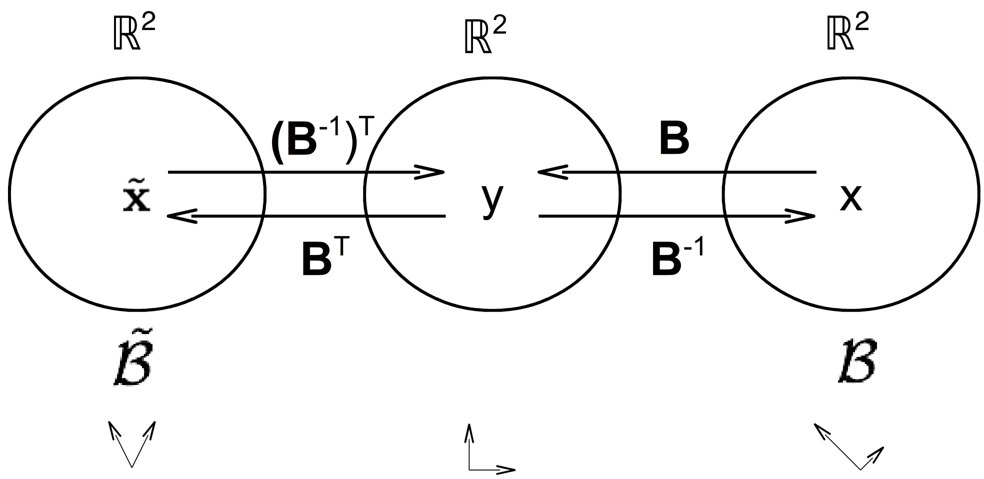

Figure 4: Primal $\mathcal{B}$ (right) and dual $\tilde{\mathcal{B}}$ (left) bases and standard basis (center). The vectors which comprise $\tilde{\mathcal{B}}$ are the transposes of the rows of $\mathbf{B}^{-1}$.

Frame Synthesis Matrix

Let $\mathcal{F}$ consist of the N frame vectors, $\mathbf{f}_1 \ldots \mathbf{f}_N \in \mathbb{R}^M$, where $N \geq M$. Let $\{\mathbf{y}\}_{\mathcal{F}} \in \mathbb{R}^N$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\mathcal{F}$. It follows that

$$\mathbf{y} = \mathbf{F}\{\mathbf{y}\}_{\mathcal{F}}$$

where the *synthesis matrix*, $\mathbf{F}$, is the $M \times N$ matrix,

$$\mathbf{F} = \left[\, \mathbf{f}_1 \mid \mathbf{f}_2 \mid \ \ldots \ \mid \mathbf{f}_N \,\right].$$

Frame Analysis Matrix

We might guess that

$$\{\mathbf{y}\}_{\mathcal{F}} = \mathbf{F}^{-1}\mathbf{y}$$

where $\mathbf{F}^{-1}$ is $N \times M$ and $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$. Unfortunately, because $\mathbf{F}$ is not square, there is no unique inverse. However, $\mathbf{F}$ has an infinite number of *right-inverses*. Each of the $\{\mathbf{y}\}_{\mathcal{F}}$ produced when $\mathbf{y}$ is multiplied by a distinct right-inverse is a distinct representation of the vector $\mathbf{y}$ in the frame, $\mathcal{F}$.

Pseudoinverse

The *pseudoinverse* of $\mathbf{F}$ is

$$\mathbf{F}^{+} = \mathbf{F}^{\mathrm{T}} \left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^{-1}.$$

$\mathbf{F}^{+}$ is a right inverse of $\mathbf{F}$ because

$$\mathbf{F}\mathbf{F}^{+} = \mathbf{F}\mathbf{F}^{\mathrm{T}} \left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^{-1} = \mathbf{I}.$$

The $N \times M$ matrix, $\mathbf{F}^{+}$, is as an *analysis matrix* because it transforms a representation $\mathbf{y} \in \mathbb{R}^{M}$, in the standard basis, into a representation $\{\mathbf{y}\}_{\mathcal{F}} \in \mathbb{R}^{N}$, in the frame, $\mathcal{F}$:

$$\{\mathbf{y}\}_{\mathcal{F}} = \mathbf{F}^{+}\mathbf{y}.$$

Dual Frame and Its Synthesis Matrix

If $\mathcal{F}$ consists of the N frame vectors, $\mathbf{f}_1 \ldots \mathbf{f}_N \in \mathbb{R}^M$, with analysis matrix $\mathbf{F}^+$, then there exists a *dual frame*, $\widetilde{\mathcal{F}}$, consisting of the N frame vectors, $\widetilde{\mathbf{f}}_1 \ldots \widetilde{\mathbf{f}}_N \in \mathbb{R}^M$:

$$\left(\mathbf{F}^+\right)^{\mathrm{T}} = \left[\widetilde{\mathbf{f}}_1 \mid \widetilde{\mathbf{f}}_2 \mid \ \ldots \ \mid \widetilde{\mathbf{f}}_N\right].$$

Let $\{\mathbf{y}\}_{\widetilde{\mathcal{F}}} \in \mathbb{R}^N$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\widetilde{\mathcal{F}}$. It follows that

$$\mathbf{y} = \left(\mathbf{F}^+\right)^{\mathrm{T}} \{\mathbf{y}\}_{\widetilde{\mathcal{F}}}.$$

and $\left(\mathbf{F}^+\right)^{\mathrm{T}}$ is the *synthesis matrix* for the dual frame, $\widetilde{\mathcal{F}}$.

Dual Frame Analysis Matrix

Because $\mathbf{F}\mathbf{F}^{+} = \mathbf{I}$, it follows that $\mathbf{F}^{\mathrm{T}}$ is a right inverse of $(\mathbf{F}^{+})^{\mathrm{T}}$:

$$(\mathbf{F}^{+})^{\mathrm{T}}\,\mathbf{F}^{\mathrm{T}} = \mathbf{I}.$$

Consequently, $\mathbf{F}^{\mathrm{T}}$ is an *analysis matrix* for the dual frame, $\widetilde{\mathcal{F}}$:

$$\{\mathbf{y}\}_{\widetilde{\mathcal{F}}} = \mathbf{F}^{\mathrm{T}}\mathbf{y}.$$

Span of Dual Frame

The N vectors $\widetilde{\mathcal{F}}$ form a frame for $\mathbb{R}^M$ iff for every $\mathbf{y} \in \mathbb{R}^M$of finite non-zero length there is a finite non-zero length representation of $\mathbf{y}$ in $\widetilde{\mathcal{F}}$:

$$A||\mathbf{y}||^2 \leq ||\{\mathbf{y}\}_{\widetilde{\mathcal{F}}}||^2 \leq B||\mathbf{y}||^2$$

where $0 < A \leq B < \infty$.

Span of Primal Frame

Because the spans of the primal ($\mathcal{F}$) and dual ($\widetilde{\mathcal{F}}$) frames are the same, and because

$$\{\mathbf{y}\}_{\widetilde{\mathcal{F}}} = \mathbf{F}^{\mathrm{T}}\mathbf{y}$$

$\mathcal{F}$ is a frame iff for all $\mathbf{y} \in \mathbb{R}^M$ there exist A and B where $0 < A \leq B < \infty$ and where

$$A||\mathbf{y}||^2 \leq ||\mathbf{F}^{\mathrm{T}}\mathbf{y}||^2 \leq B||\mathbf{y}||^2.$$

A and B are called the *frame bounds*. This is significant because this is a necessary and sufficient condition for a set of vectors (the columns of $\mathbf{F}$) to form a frame.

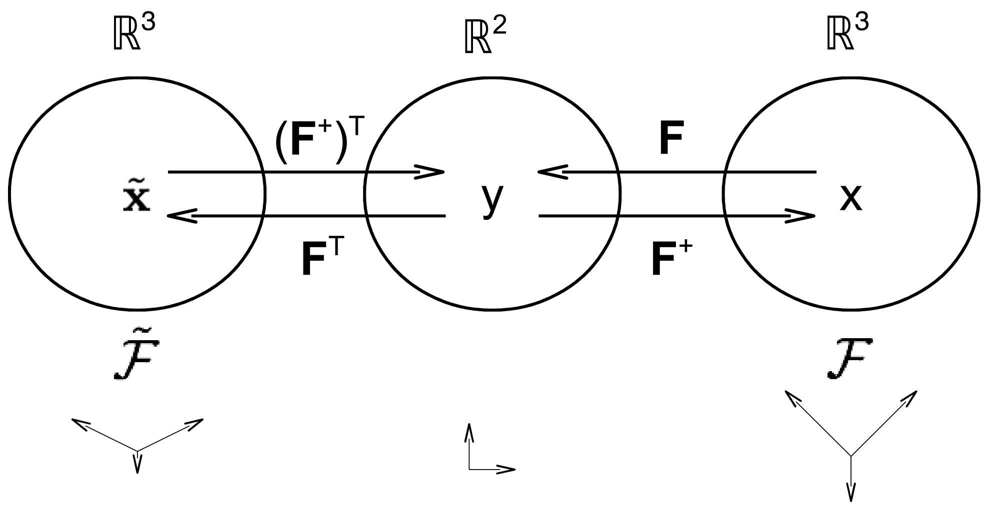

Figure 5: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) frames and standard basis (center). The vectors which comprise $\tilde{\mathcal{F}}$ are the transposes of the rows of $\mathbf{F}^+$.

Example

What is the representation of $\mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ in the frame formed by the vectors $\mathbf{f}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$, $\mathbf{f}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$ and $\mathbf{f}_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$?

$$\mathbf{F} = \begin{bmatrix} 0.70711 & -0.70711 & 0 \\ 0.70711 & 0.70711 & -1 \end{bmatrix}$$

$$\mathbf{F}^{+} = \begin{bmatrix} 0.70711 & 0.35355 \\ -0.70711 & 0.35355 \\ 0 & -0.5 \end{bmatrix}$$

$$\mathbf{F}^{+}\mathbf{y} = \begin{bmatrix} 1.06066 \\ -0.35355 \\ -0.5 \end{bmatrix}$$

Tight-Frames

If $A = B$ then

$$||\mathbf{F}^{\mathrm{T}}\mathbf{y}||^2 = A||\mathbf{y}||^2$$

and $\mathcal{F}$ is said to be a *tight-frame*. When $\mathcal{F}$ is a tight-frame,

$$\mathbf{F}^{+} = \frac{1}{A}\mathbf{F}^{\mathrm{T}}.$$

If $||\mathbf{f}_i|| = 1$ for all frame vectors, $\mathbf{f}_i$, then A equals the *overcompleteness* of the representation. When $A = B = 1$, then $\mathcal{F}$ is an *orthonormal basis* and $\mathcal{F} = \widetilde{\mathcal{F}}$.

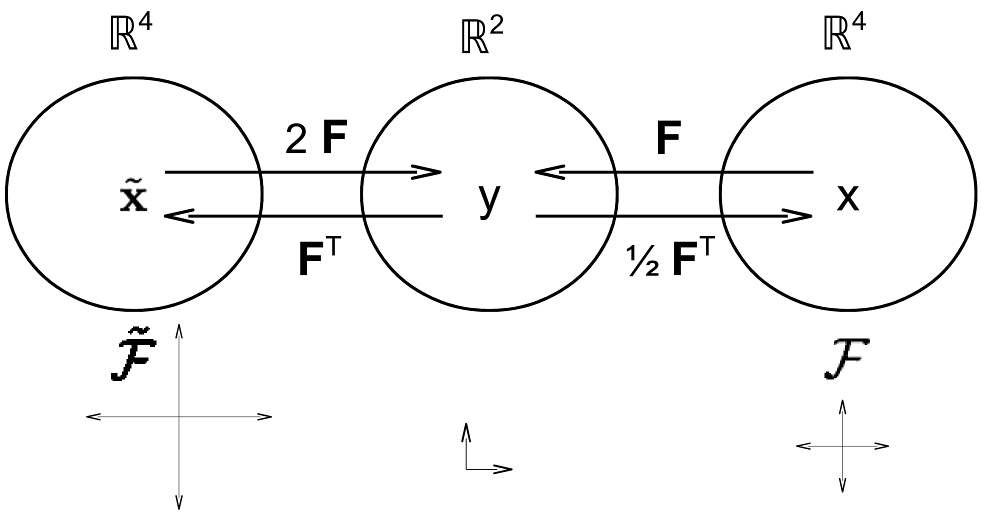

Figure 6: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness two and standard basis (center).

Example

What is the representation of $\mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$ in the frame formed by the vectors $\mathbf{f}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f}_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{f}_4 = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}$?

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{F}^{+} = \frac{1}{2}\mathbf{F}^{\mathrm{T}} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \\ 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}$$

$$\frac{1}{2}\mathbf{F}^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

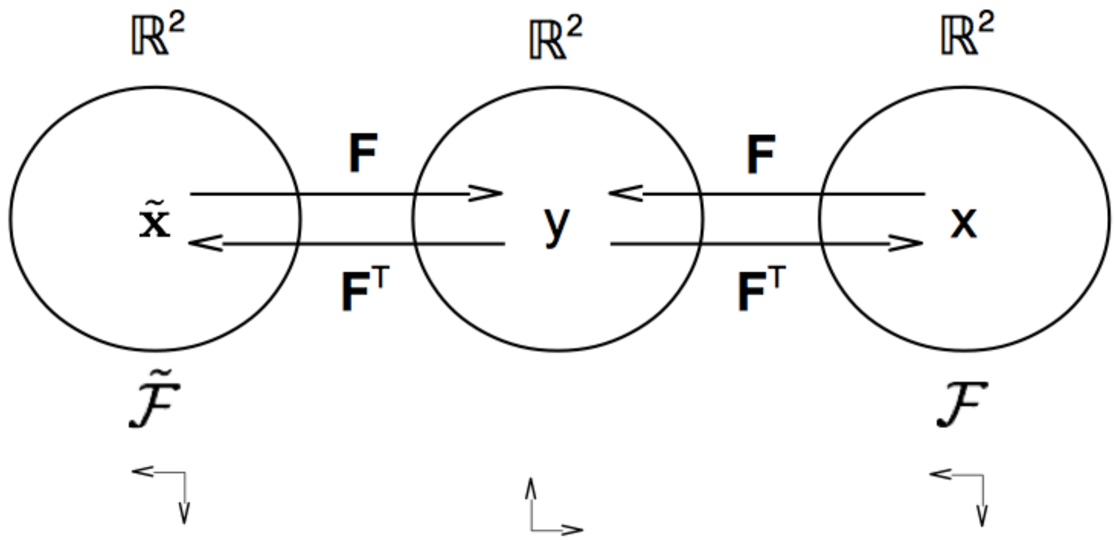

Figure 7: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness one (orthonormal bases) and standard basis (center).

Summary of Notation

- $\mathbf{y} \in \mathbb{R}^M$ – a vector.
- $\{\mathbf{y}\}_{\mathcal{F}} \in \mathbb{R}^N$ – a representation of $\mathbf{y}$ in primal frame $\mathcal{F}$.
- $\mathbf{f}_1 \ldots \mathbf{f}_N \in \mathbb{R}^M$ where $N \geq M$ – frame vectors for primal frame $\mathcal{F}$.
- $\mathbf{F} = \left[\, \mathbf{f}_1 \mid \mathbf{f}_2 \mid \ \ldots \ \mid \mathbf{f}_N \,\right]$ – synthesis matrix for primal frame $\mathcal{F}$.
- $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^M$.
- $\mathbf{F}^+ = \mathbf{F}^{\mathrm{T}} \left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)^{-1}$ – analysis matrix for primal frame $\mathcal{F}$.
- $\mathbf{F}^+ : \mathbb{R}^M \to \mathbb{R}^N$.
- $0 < A \leq B < \infty$ – bounds for primal frame $\mathcal{F}$.

Summary of Notation (contd.)

- $\{\mathbf{y}\}_{\widetilde{\mathcal{F}}} \in \mathbb{R}^M$ – a representation of $\mathbf{y}$ in dual frame $\widetilde{\mathcal{F}}$.
- $\widetilde{\mathbf{f}}_1 \ldots \widetilde{\mathbf{f}}_N \in \mathbb{R}^M$ – frame vectors for dual frame $\widetilde{\mathcal{F}}$.
- $\left(\mathbf{F}^{+}\right)^{\mathrm{T}} = \left[\widetilde{\mathbf{f}}_1 \mid \widetilde{\mathbf{f}}_2 \mid \ \ldots \ \mid \widetilde{\mathbf{f}}_N\right]$ – synthesis matrix for dual frame $\widetilde{\mathcal{F}}$.
- $\left(\mathbf{F}^{+}\right)^{\mathrm{T}} : \mathbb{R}^N \to \mathbb{R}^M$.
- $\mathbf{F}^{\mathrm{T}}$ – analysis matrix for dual frame $\widetilde{\mathcal{F}}$.
- $\mathbf{F}^{\mathrm{T}} : \mathbb{R}^M \to \mathbb{R}^N$.
- $0 < \frac{1}{B} \leq \frac{1}{A} < \infty$ – bounds for dual frame $\widetilde{\mathcal{F}}$.