The 2D Fourier Transform

The analysis and synthesis formulas for the 2D continuous Fourier transform are as follows:

- **Analysis**

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dx\,dy$$

- **Synthesis**

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}du\,dv$$

The 2D Frequency Domain

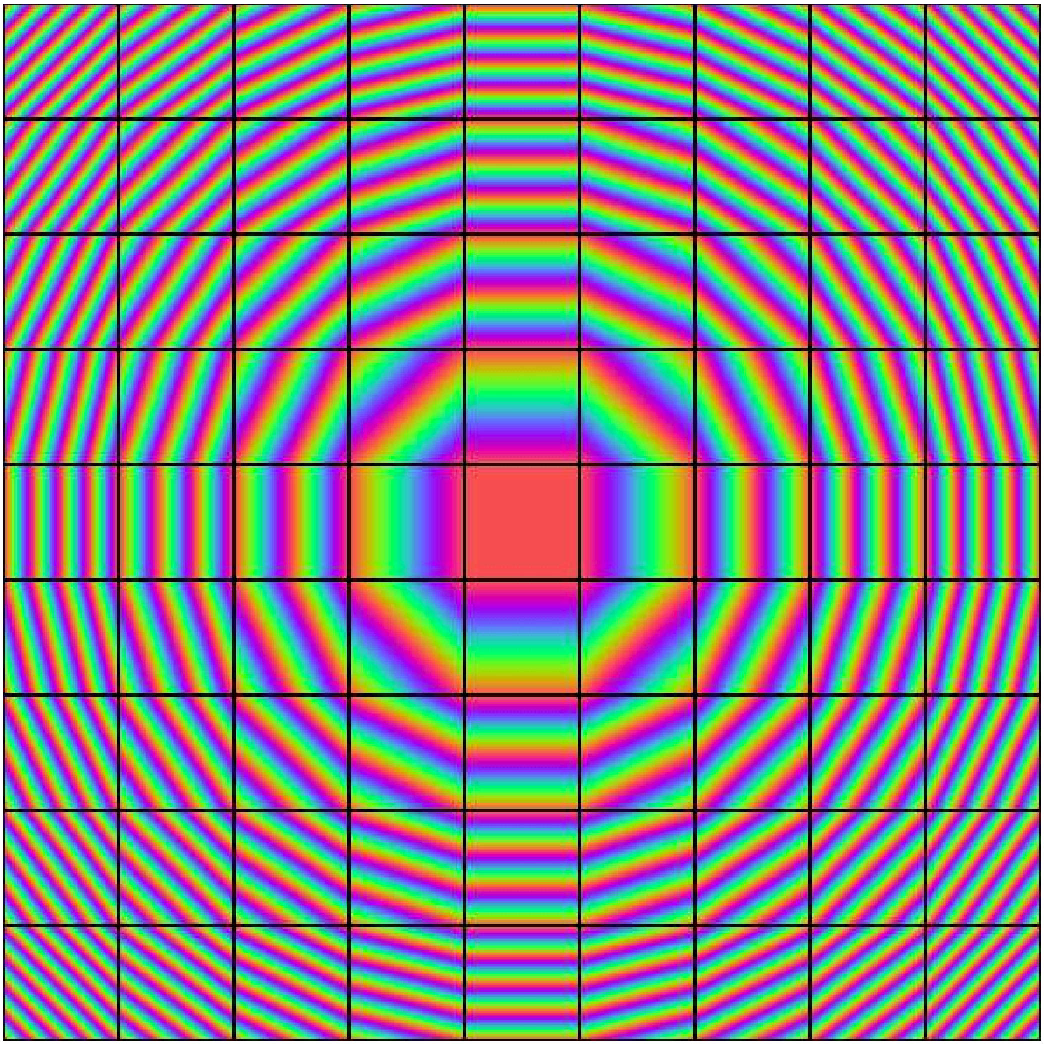

Figure 1: A 9×9 table of harmonic grating basis functions, $e^{j2\pi(ux+vy)}$, where $u \in \{-4 \ldots 4\}$ and $v \in \{-4 \ldots 4\}$. Color indicates phase with red signifying real positive, green signifying imaginary positive, blue signifying real negative and violet signifying imaginary negative.

Separability of 2D Fourier Transform

The 2D analysis formula can be written as a 1D analysis in the x direction followed by a 1D analysis in the y direction:

$$F(u,v) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi ux} dx \right] e^{-j2\pi vy} dy.$$

The 2D synthesis formula can be written as a 1D synthesis in the u direction followed by a 1D synthesis in v direction:

$$f(x,y) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(u,v) e^{j2\pi ux} du \right] e^{j2\pi vy} dv.$$

Separability Theorem

$$f(x,y) = f(x)g(y) \xrightarrow{\mathcal{F}} F(u,v) = F(u)G(v)$$

Proof:

$$F(u,v)$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-j2\pi ux}e^{-j2\pi vy}\,dx\,dy \\
&= \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}\,dx \int_{-\infty}^{\infty} g(y)e^{-j2\pi vy}dy \\
&= F(u)\,G(v)
\end{aligned}$$

The 2D Discrete Fourier Transform

The analysis and synthesis formulas for the 2D discrete Fourier transform are as follows:

- **Analysis**

$$\hat{F}(k,\ell) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F(m,n) e^{-j2\pi\left(k\frac{m}{M}+\ell\frac{n}{N}\right)}$$

- **Synthesis**

$$F(m,n) = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{\ell=0}^{N-1} \hat{F}(k,\ell) e^{\,j2\pi\left(k\frac{m}{M}+\ell\frac{n}{N}\right)}$$

Separability of the 2D DFT

$$\hat{F}(k,\ell) =$$
$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\left[\frac{1}{\sqrt{M}}\sum_{m=0}^{M-1}F(m,n)e^{-j2\pi\left(k\frac{m}{M}\right)}\right]e^{-j2\pi\left(\ell\frac{n}{N}\right)}$$

The 2D forward DFT can be written in matrix notation:

$$\hat{\mathbf{F}} = \left(\mathbf{W}^{*}\mathbf{F}\right)\mathbf{W}^{*}$$

where

$$W^{*}_{mn} = \frac{1}{\sqrt{N}}e^{-j2\pi m\frac{n}{N}}$$

and

$$F(m,n) =$$
$$\frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}\left[\frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}\hat{F}(k,\ell)e^{\,j2\pi\left(k\frac{m}{M}\right)}\right]e^{\,j2\pi\left(\ell\frac{n}{N}\right)}.$$

Separability of the 2D DFT (contd.)

The 2D inverse DFT can be written in matrix notation:

$$\mathbf{F} = \left(\mathbf{W}\hat{\mathbf{F}}\right)\mathbf{W}$$

where

$$W_{mn} = \frac{1}{\sqrt{N}} e^{\,j2\pi m \frac{n}{N}}.$$