Convolution with Highpass Filter

Convolution with the Haar highpass filter is implemented as multiplication with a circulant matrix:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

where u_i are the components of the upper half-band signal and x_i are the components of the input signal.

Convolution with Lowpass Filter

Convolution with the Haar lowpass filter is also implemented as multiplication with a circulant matrix:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

where v_i are the components of the lower half-band signal and x_i are the components of the input signal.

Downsampling the Upper Half-band

The factor of two downsampling of the upper half-band signal is accomplished by discarding the even numbered rows. This forms the **U** matrix:

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

Downsampling the Lower Half-band

The factor of two downsampling of the lower half-band signal is accomplished by discarding the even numbered rows. This forms the **L** matrix:

$$\begin{bmatrix} v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

Two Channel Encoding Matrix

By combining **U** and **L** into one matrix, we create a two channel encoding matrix:

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}.$$

Written using block matrix notation:

$$\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{v} \end{array}\right] = \frac{1}{\sqrt{2}} \left[\begin{array}{c} \mathbf{U} \\ \hline \mathbf{L} \end{array}\right] \mathbf{x}.$$

Two Channel Decoding Matrix

The two channel decoding matrix is the inverse of the two channel encoding matrix. Because the rows of the two channel encoding matrix are orthogonal, its inverse is just its transpose:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix}.$$

Written using block matrix notation:

$$\mathbf{x} = \frac{1}{\sqrt{2}} \left[\, \mathbf{U}^{\mathrm{T}} \,\middle|\, \mathbf{L}^{\mathrm{T}} \,\right] \left[\frac{\mathbf{u}}{\mathbf{v}} \right].$$

Second Stage Encoding Matrix

We can create a three channel encoder by using a two channel encoder to encode the lower half-band signal:

$$\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ u'_1 \\ u'_3 \\ v'_1 \\ v'_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ v_1 \\ v_3 \\ v_5 \\ v_7 \end{bmatrix}.$$

Second Stage Encoding Matrix (contd.)

Written in block matrix notation:

$$\left[\begin{array}{c}\mathbf{u}\\ \hline \mathbf{u}'\\ \hline \mathbf{v}'\end{array}\right] = \left[\begin{array}{c|c}\mathbf{I}' & \mathbf{0}\\ \hline \mathbf{0} & \frac{1}{\sqrt{2}}\begin{array}{c}\mathbf{U}'\\ \hline \mathbf{L}'\end{array}\end{array}\right]\left[\begin{array}{c}\mathbf{u}\\ \hline \mathbf{v}\end{array}\right]$$

The upper left quadrant of the second stage encoding matrix is a half-sized identity matrix; this passes the upper half-band signal unchanged. The lower right quadrant is a half-sized two channel encoding matrix; this encodes the lower half-band signal.

Three Channel Encoding Matrix

Putting the two stages together yields:

$$\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{u}' \\ \hline \mathbf{v}' \end{array}\right] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \mathbf{I}' & \mathbf{0} \\ \hline \mathbf{0} & \frac{1}{\sqrt{2}} \begin{array}{c} \mathbf{U}' \\ \hline \mathbf{L}' \end{array} \end{array}\right] \left[\begin{array}{c} \mathbf{U} \\ \hline \mathbf{L} \end{array}\right] \mathbf{x}.$$

Haar Matrix

This pattern can be abstracted to derive a recursive definition for an $N+1$ channel encoding matrix:

$$\mathbf{H}_N = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \mathbf{I}_{N-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{H}_{N-1} \end{array} \right] \left[\begin{array}{c} \mathbf{U}_N \\ \hline \mathbf{L}_N \end{array} \right]$$

where $\mathbf{U}_N$ convolves a length 2^N signal with the Haar highpass filter followed by downsampling, $\mathbf{L}_N$ convolves a length 2^N signal with the Haar lowpass filter followed by downsampling, $\mathbf{I}_N$ is the identity matrix of size $2^N \times 2^N$ and

$$\mathbf{H}_1 = \frac{1}{\sqrt{2}} \left[\begin{array}{c} \mathbf{U}_1 \\ \hline \mathbf{L}_1 \end{array} \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Haar Matrix Example

$$\left[\frac{\mathbf{U}_2}{\mathbf{L}_2}\right] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{H}_2 = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Inverse Haar Matrix

Because the Haar matrix is unitary, the $N+1$ channel decoding matrix is simply the transpose of the $N+1$ channel encoding matrix:

$$\mathbf{H}_N^{\mathrm{T}} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \mathbf{U}_N^{\mathrm{T}} & \mathbf{L}_N^{\mathrm{T}} \end{array} \right] \left[\begin{array}{c|c} \mathbf{I}_{N-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{H}_{N-1}^{\mathrm{T}} \end{array} \right].$$

Inverse Haar Matrix Example

$$\left[\, \mathbf{U}_2^{\mathrm{T}} \mid \mathbf{L}_2^{\mathrm{T}} \,\right] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2^{\mathrm{T}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

It is easy to verify that

$$\mathbf{H}_2 \mathbf{H}_2^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

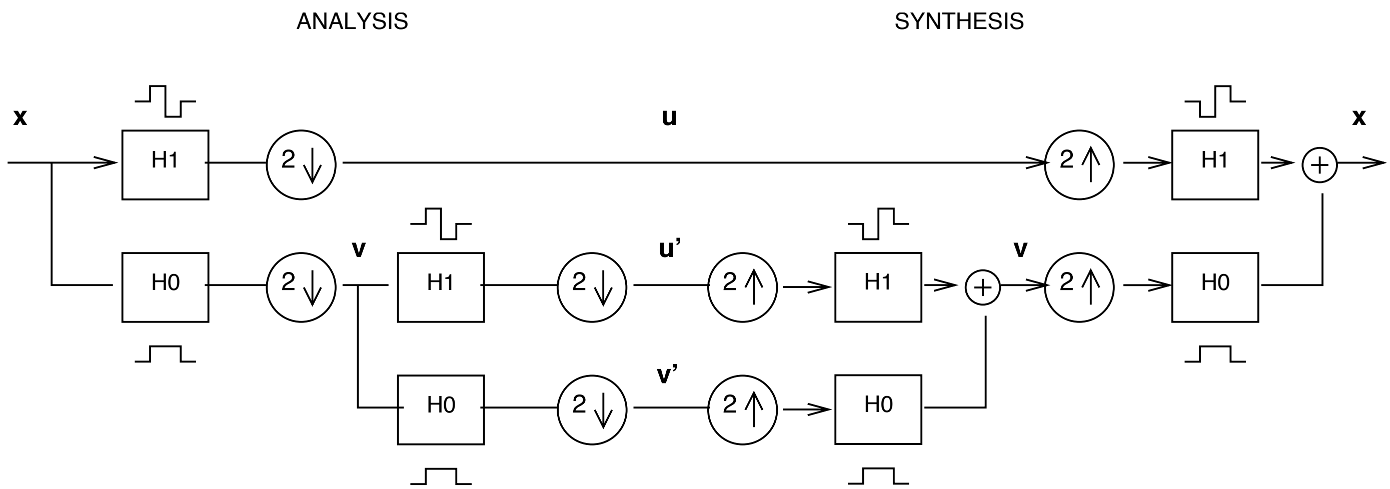

Figure 1: Three channel Haar transform.