CS 530: Geometric and Probabilistic Methods in Computer Science
Homework 5 (Fall '15)

1. Let $f(t) = e^{-\pi t^2}$, $f''(t) = 2\pi e^{-\pi t^2}(2\pi t^2 - 1)$, and $g(t) = at + b$. Prove or disprove the following:
$$\langle f'', g \rangle = 0$$
for all a and b.

2. Let $\Psi(t) = e^{-\pi t^2}\cos(2\pi s_0 t)$ and $f(t) = e^{j2\pi s_1 t}$. Give an expression for $F(a,b)$, the continuous wavelet transform of $f(t)$.

3. The n-th moment of Ψ is defined to be $M_n\{\Psi\} = \int_{-\infty}^{\infty} t^n \Psi(t)dt$. Let $f(t) = e^{-\pi t^2}$, $f'(t) = -2\pi t e^{-\pi t^2}$, and $f''(t) = 2\pi e^{-\pi t^2}(2\pi t^2 - 1)$. Prove the following:
   (a) $M_0\{f'\} = 0$.
   (b) $M_0\{f''\} = M_1\{f''\} = 0$.

4. The six vectors, $\mathbf{f}_1 = \left[\begin{array}{cc} \cos(\pi/3) & \sin(\pi/3) \end{array} \right]^{\mathrm{T}}$, $\mathbf{f}_2 = \left[\begin{array}{cc} \cos(\pi/3) & -\sin(\pi/3) \end{array} \right]^{\mathrm{T}}$, $\mathbf{f}_3 = \left[\begin{array}{cc} -1 & 0 \end{array} \right]^{\mathrm{T}}$, $\mathbf{f}_4 = \left[\begin{array}{cc} -\cos(\pi/3) & -\sin(\pi/3) \end{array} \right]^{\mathrm{T}}$, $\mathbf{f}_5 = \left[\begin{array}{cc} -\cos(\pi/3) & \sin(\pi/3) \end{array} \right]^{\mathrm{T}}$, and $\mathbf{f}_6 = \left[\begin{array}{cc} 1 & 0 \end{array} \right]^{\mathrm{T}}$ form a frame $\mathcal{F}$ for $\mathbb{R}^2$. Draw the frame.
   (a) Give two representations for the vector, $\mathbf{x} = \left[\begin{array}{cc} 1 & 1 \end{array} \right]^{\mathrm{T}}$, in $\mathcal{F}$
   (b) Prove that $\mathbf{x}$ has an infinite number of representations in $\mathcal{F}$.
   (c) Give a matrix which transforms any representation of a vector in $\mathcal{F}$ into its representation in the standard basis for $\mathbb{R}^2$.
   (d) Give a matrix which transforms a representation of any vector in the standard basis for $\mathbb{R}^2$ into its representation in $\mathcal{F}$.

5. The continuous representation of the Haar highpass filter is
$$h_1(t) = \frac{1}{2}[\delta(t + \Delta t) - \delta(t - \Delta t)].$$
The continuous representation of the Haar lowpass filter is
$$h_0(t) = \frac{1}{2}[\delta(t + \Delta t) + \delta(t - \Delta t)].$$

Prove that

$$|H_0(s)|^2 + |H_1(s)|^2 = 1$$

where $H_0(s)$ and $H_1(s)$ are the Fourier transforms of $h_0(t)$ and $h_1(t)$.

6. The $N+1$ channel Haar transform matrix can be recursively defined as follows:

$$\mathbf{H}_N = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \mathbf{I}_{N-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{H}_{N-1} \end{array} \right] \left[\begin{array}{c} \mathbf{U}_N \\ \hline \mathbf{L}_N \end{array} \right]$$

where $\mathbf{U}_N$ convolves a length 2^N signal with the Haar highpass filter followed by downsampling, $\mathbf{L}_N$ convolves a length 2^N signal with the Haar lowpass filter followed by downsampling, $\mathbf{I}_N$ is the identity matrix of size $2^N \times 2^N$ and

$$\mathbf{H}_1 = \frac{1}{\sqrt{2}} \left[\begin{array}{c} \mathbf{U}_1 \\ \hline \mathbf{L}_1 \end{array} \right] = \frac{1}{\sqrt{2}} \left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right].$$

(a) Using the above definitions, derive expressions for $\mathbf{H}_3$ and $\mathbf{H}_3^{-1}$.

(b) Compute the Haar transform of the vector $\left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right]^{\mathrm{T}}$.