Markov Random Fields

- A lattice S with sites s.
- A random variable X_s ranging over a set of values V associated with each site in the lattice.
- A realization x_s of the r.v. X_s.
- The set of neighbors N_s of site s in the lattice.
- A conditional p.m.f.:

$$P(X_s = x_s \mid X_t = x_t,\ t \in N_s).$$

Markov Random Fields (contd.)

- A joint r.v. X ranging over all possible lattice configurations.
- A realization x of the joint r.v. X representing a specific lattice configuration.
- A joint p.m.f.:

$$P(X = x).$$

(a)

(b)

(c)

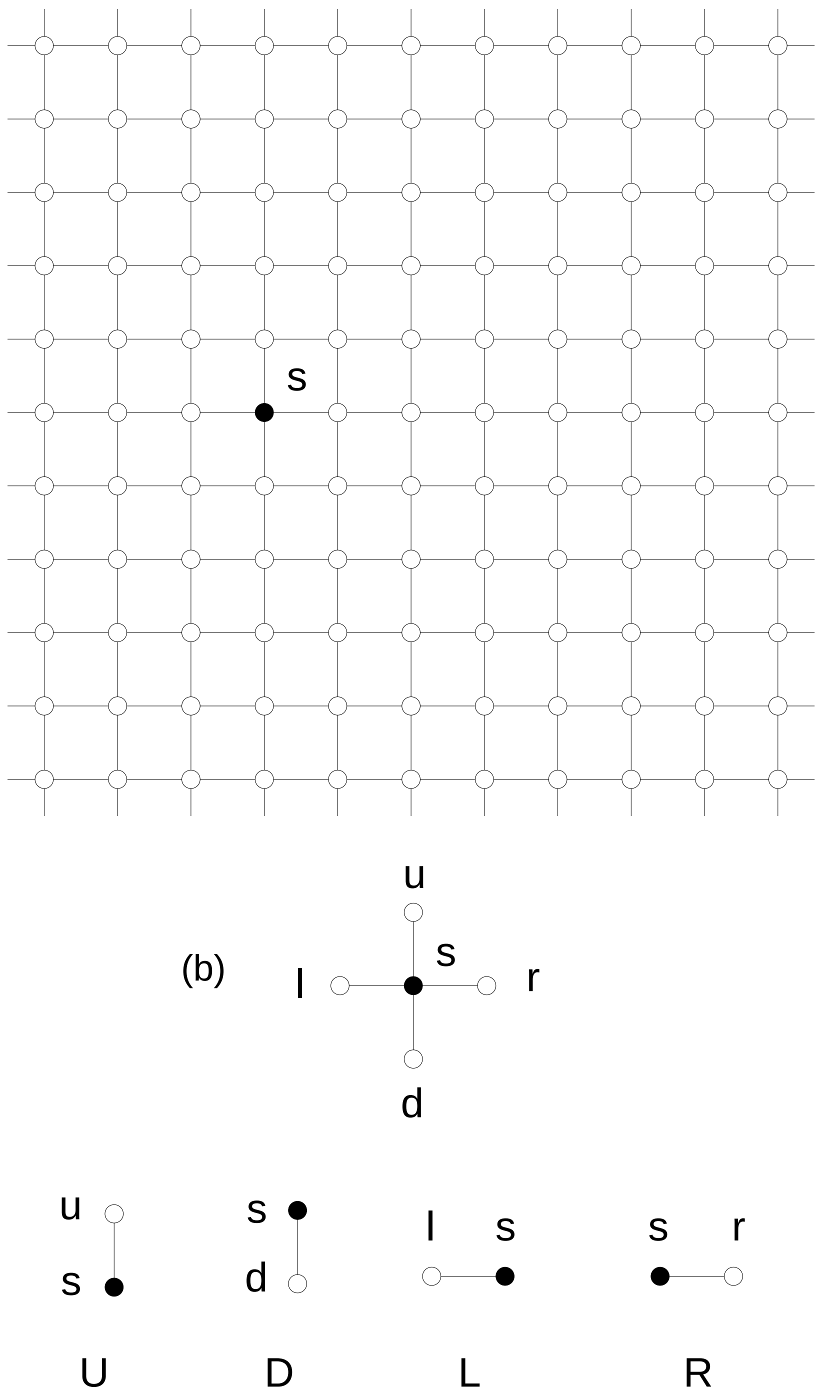

Figure 1: (a) A lattice, S. (b) A site s (filled circle) and its neighbors $N_s = \{u,d,l,r\}$ (unfilled circles) used in first-order Ising model. (c) Clique set $C_s = \{U,D,L,R\}$ used in first-order Ising model.

Markov Random Fields (contd.)

The Markov property can be defined as follows:

$$P(X_s = x_s \mid X_t = x_t,\, t \neq s,\, t \in S)$$
$$= P(X_s = x_s \mid X_t = x_t,\, t \in N_s)$$

where

$$P(X_s = x_s \mid X_t = x_t,\, t \in N_s)$$
$$= \frac{P(X_s = x_s, X_t = x_t,\, t \in N_s)}{P(X_t = x_t,\, t \in N_s)}$$

and where the above conditional p.m.f. is the same for all s.

Gibbs Sampler

To generate a sample from the joint p.m.f., $P(X=x)$, we can use a procedure called the *Gibbs sampler*:

1. Choose $s \in S$ at random.
2. Replace X_s with a sample x_s drawn from:

$$P(X_s = x_s \mid X_t = x_t,\ t \in N_s).$$

3. Repeat many times.

Markov-Gibbs equivalence

A *clique* C is a subset of the lattice S which satisfies either of the following conditions:

- C consists of a single site
- Every pair of distinct sites in C are neighbors, *i.e.*, if $s, r \in C$ and $s \neq r$ then $s \in N_r$ and $r \in N_s$.

Given this definition of clique, we can define C_s, the *local clique set* for site s, to be the set of cliques of S which contain s.

Markov-Gibbs equivalence (contd.)

The *clique potential function* V_C for clique $C \in C_s$ is defined as follows:

$$V_C(x_s\,;\,x_t,\,t \neq s, t \in C) =$$

$$\ln P(X_s = x_s \mid X_t = x_t,\,t \neq s, t \in C).$$

Markov-Gibbs equivalence (contd.)

It can be shown that the conditional p.m.f. for any MRF can be written in the following form:

$$P(X_s = x_s \mid X_t = x_t,\, t \in N_s) = \frac{\exp[-\sum_{C \in C_s} V_C(x_s\,;\, x_t,\, t \neq s, t \in C)]}{\sum_{y_s \in V} \exp[-\sum_{C \in C_s} V_C(y_s\,;\, x_t,\, t \neq s, t \in C)]}.$$

A MRF defined this way is called a *Gibb's Random Field*.

Ising Model

The *Ising model* is a standard model of the emergence of spatial organization in ferromagnetic materials. We assume that each site s in a rectangular lattice can possess one of two spins:

$$X_s \in \{+1, -1\}.$$

The neighborhood set of site s is $N_s = \{u, d, l, r\}$. The clique set, $C_s = \{U, D, L, R\}$, contains four cliques of size two: $U = \{s, u\}$, $D = \{s, d\}$, $L = \{s, l\}$ and $R = \{s, r\}$.

Ising Model (contd.)

In the Ising model, the clique potential functions are:

$$V_U(x,y) = V_D(x,y) = V_L(x,y) = V_R(x,y) = -xy.$$

Consequently, the conditional p.m.f. is

$$P(X_s = x_s \mid X_t = x_t \in N_s) =$$

$$\frac{\exp\left[x_s(x_u + x_d + x_l + x_r)\right]}{\sum_{y_s \in \{+1,-1\}} \exp\left[y_s(x_u + x_d + x_l + x_r)\right]}.$$

Gibbs Sampling in the Ising Model

$$P\left(\begin{array}{ccc} & +1 & \\ -1 & \cdot & -1 \\ & +1 & \end{array} \longrightarrow \begin{array}{ccc} & +1 & \\ -1 & +1 & -1 \\ & +1 & \end{array}\right) =$$

$$P(X_s = +1 \mid X_u = +1, X_d = +1, X_l = -1, X_r = -1) =$$

$$\frac{\exp\left[+1(1+1-1-1)\right]}{\exp\left[+1(1+1-1-1)\right] + \exp\left[-1(1+1-1-1)\right]} = 0.5$$

$$P\left(\begin{array}{ccc} & +1 & \\ -1 & \cdot & -1 \\ & +1 & \end{array} \longrightarrow \begin{array}{ccc} & +1 & \\ -1 & -1 & -1 \\ & +1 & \end{array}\right) =$$

$$P(X_s = -1 \mid X_u = +1, X_d = +1, X_l = -1, X_r = -1) =$$

$$\frac{\exp\left[-1(1+1-1-1)\right]}{\exp\left[+1(1+1-1-1)\right] + \exp\left[-1(1+1-1-1)\right]} = 0.5$$

Gibbs Sampling in the Ising Model (contd.)

$$P\left(\begin{array}{ccc} & +1 & \\ +1 & \cdot & -1 \\ & +1 & \end{array} \longrightarrow \begin{array}{ccc} & +1 & \\ +1 & +1 & -1 \\ & +1 & \end{array}\right) =$$

$$P(X_s = +1 \mid X_u = +1, X_d = +1, X_l = +1, X_r = -1) =$$

$$\frac{\exp\left[+1(1+1+1-1)\right]}{\exp\left[+1(1+1+1-1)\right] + \exp\left[-1(1+1+1-1)\right]} \approx 0.98$$

$$P\left(\begin{array}{ccc} & +1 & \\ +1 & \cdot & -1 \\ & +1 & \end{array} \longrightarrow \begin{array}{ccc} & +1 & \\ +1 & -1 & -1 \\ & +1 & \end{array}\right) =$$

$$P(X_s = -1 \mid x_u = +1, x_d = +1, x_l = +1, x_r = -1) =$$

$$\frac{\exp\left[-1(1+1+1-1)\right]}{\exp\left[+1(1+1+1-1)\right] + \exp\left[-1(1+1+1-1)\right]} \approx 0.02$$

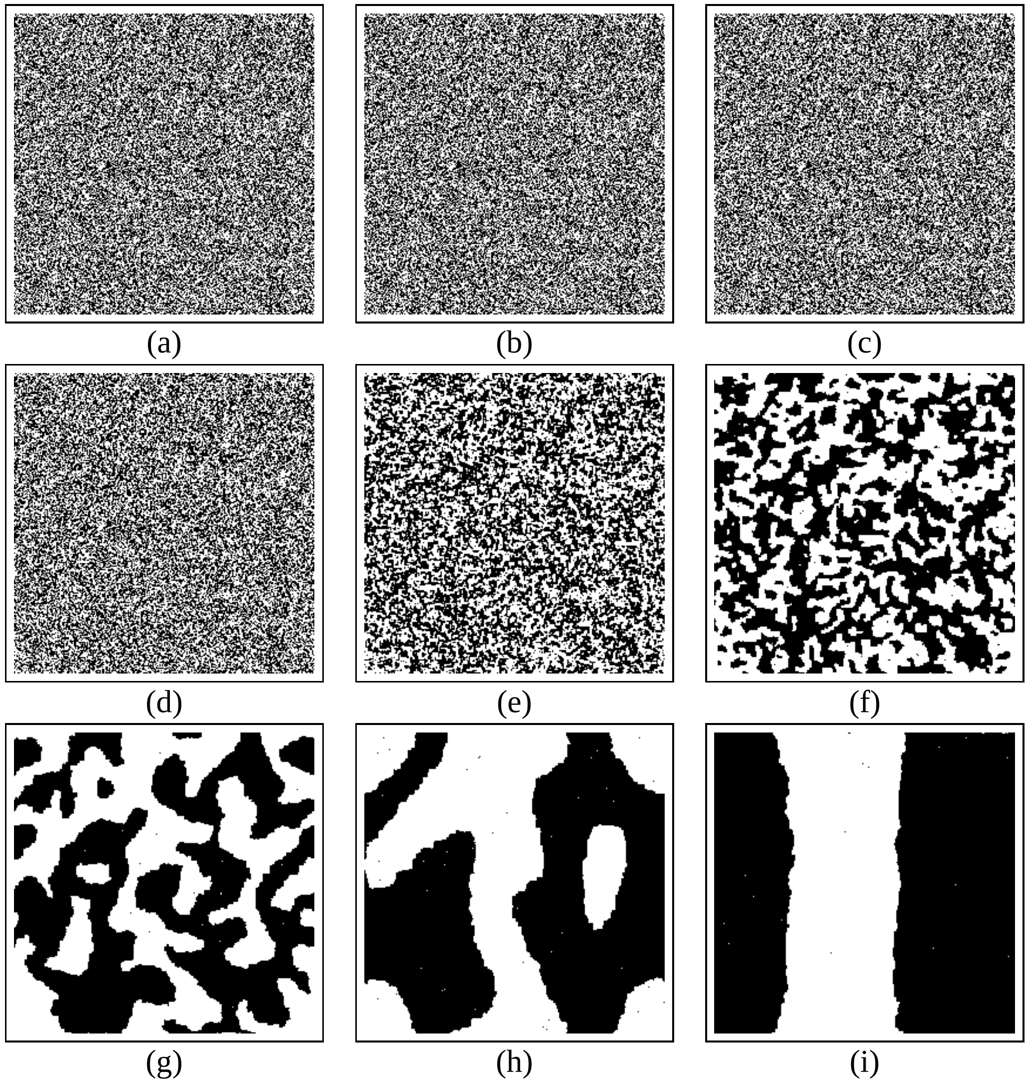

Figure 2: Ising model. (a) Initial configuration of 256×256 toroidal lattice. (b) After 10^2 iterations of Gibbs sampling. (c) After 10^3 iterations. (d) After 10^4 iterations. (e) After 10^5 iterations. (f) After 10^6 iterations. (g) After 10^7 iterations. (h) After 10^8 iterations. (i) After 10^9 iterations.