Scaled Shah Function

Recall that the *Shah function* is defined as

$$\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

and that the impulse behaves as follows when scaled:

$$\begin{aligned} |a|\delta(at) &= \delta(t) \\ \delta(at) &= \frac{1}{|a|}\delta(t). \end{aligned}$$

Using the above we can derive an expression for a scaled Shah function:

$$\begin{aligned} \text{III}(at) &= \sum_{n=-\infty}^{\infty} \delta(at-n) \\ &= \sum_{n=-\infty}^{\infty} \delta\left[a\left(t-\frac{n}{a}\right)\right] \\ &= \frac{1}{|a|}\sum_{n=-\infty}^{\infty} \delta\left(t-\frac{n}{a}\right). \end{aligned}$$

Scaled Shah Function (contd.)

When $a = 1/\tau$ and $\tau > 0$:

$$\text{III}\left(t/\tau\right) = \tau \sum_{n=-\infty}^{\infty} \delta(t - n\tau).$$

Dividing both sides by τ results in a Shah function composed of impulses of unit magnitude spaced τ apart:

$$\text{III}\left(t/\tau\right)/\tau = \sum_{n=-\infty}^{\infty} \delta(t - n\tau).$$

Fourier Transform of the Shah Function

Recall that the Shah function is its own Fourier transform:

$$\mathcal{F}\{\mathrm{III}\} = \mathrm{III}.$$

We can use the Similarity Theorem

$$\mathcal{F}\{f(at)\}(s) = \frac{1}{|a|}F\left(\frac{s}{a}\right)$$

to derive an expression for the Fourier transform of a Shah function composed of impulses of unit magnitude spaced τ apart:

$$\mathcal{F}\{\mathrm{III}(t/\tau)/\tau\}(s) = \mathrm{III}(\tau s)$$

where

$$\mathrm{III}(\tau \mathrm{s}) = \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\delta(s-n/\tau).$$

This is a Shah function composed of impulses of magnitude $\frac{1}{\tau}$ spaced $\frac{1}{\tau}$ apart.

Sampling Theorem

Let f_d be a discretely sampled version of f:

$$\begin{aligned} f_d(t) &= \sum_{n=-\infty}^{\infty} f(t)\delta(t-n\tau) \\ &= f(t) \sum_{n=-\infty}^{\infty} \delta(t-n\tau). \end{aligned}$$

Since

$$\mathrm{III}(t/\tau)/\tau = \sum_{n=-\infty}^{\infty} \delta(t-n\tau)$$

it follows that:

$$f_d(t) = f(t) \cdot \mathrm{III}\left(t/\tau\right)/\tau.$$

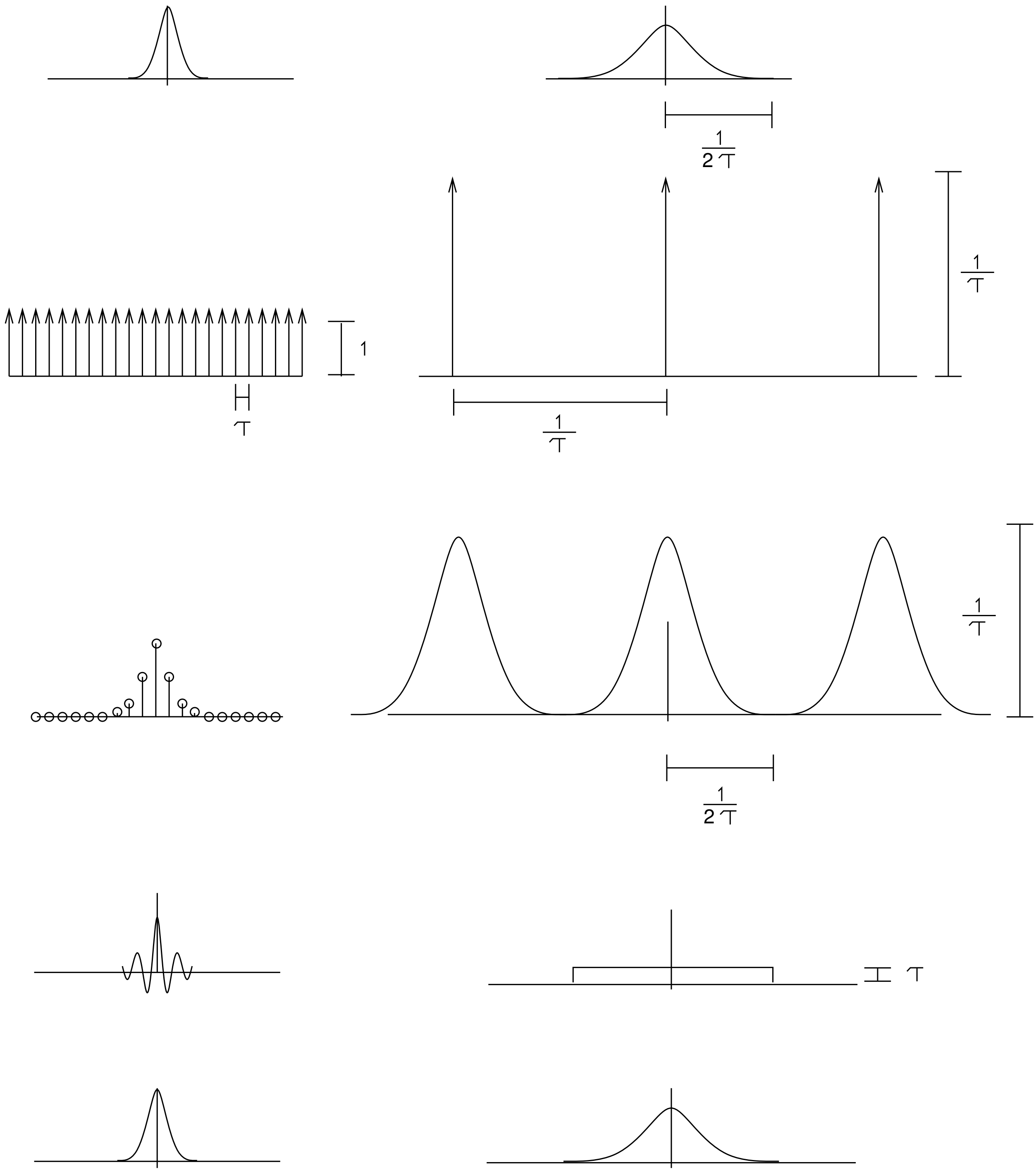

Figure 1: Reconstructing a continuous signal from discrete samples. Time domain (left). Frequency domain (right).

Sampling Theorem (contd.)

- **Question** For what values of τ can we recover f from f_d?
- **Idea** Perhaps it's easier to recover F from F_d. Taking the Fourier transform of both sides of

$$f_d(t) = f(t) \cdot \mathrm{III}\left(t/\tau\right)/\tau$$

yields

$$\begin{aligned}
\mathcal{F}\left\{f_d(t)\right\}(s) &= \mathcal{F}\left\{f(t)\right\}(s) * \mathcal{F}\left\{\mathrm{III}(t/\tau)/\tau\right\}(s) \\
F_d(s) &= F(s) * \mathrm{III}\left(\tau s\right) \\
&= F(s) * \frac{1}{\tau} \sum_{n=-\infty}^{\infty} \delta(s - n/\tau) \\
&= \frac{1}{\tau} \sum_{n=-\infty}^{\infty} F(s) * \delta(s - n/\tau) \\
&= \frac{1}{\tau} \sum_{n=-\infty}^{\infty} F\left(s - n/\tau\right).
\end{aligned}$$

Sampling Theorem (contd.)

Let $F(s) = 0$ for $|s| \geq s_1$. If $0 < \tau < 1/(2s_1)$, then multiplying both sides by an ideal lowpass filter, $\tau\Pi(\tau s)$, yields:

$$\begin{aligned} \tau\Pi(\tau s) \cdot F_d(s) &= \Pi(\tau s) \sum_{n=-\infty}^{\infty} F(s - n/\tau) \\ &= F(s). \end{aligned}$$

What does this look like in the time domain? Applying $\mathcal{F}^{-1}$ to both sides of

$$\tau\Pi(\tau s) \cdot F_d(s) = F(s)$$

yields

$$\begin{aligned} \mathcal{F}^{-1}\{\tau\Pi(\tau s)\}(t) * \mathcal{F}^{-1}\{F_d(s)\}(t) &= \mathcal{F}^{-1}\{F(s)\}(t) \\ \frac{\sin(\pi t/\tau)}{(\pi t/\tau)} * f_d(t) &= f(t). \end{aligned}$$

Useful jargon: $\sin(\pi t/\tau)/(\pi t/\tau)$ is called the *sinc* function, s_1 is called the *folding frequency*, and $\tau = 1/(2s_1)$ is called the *Nyquist rate*.

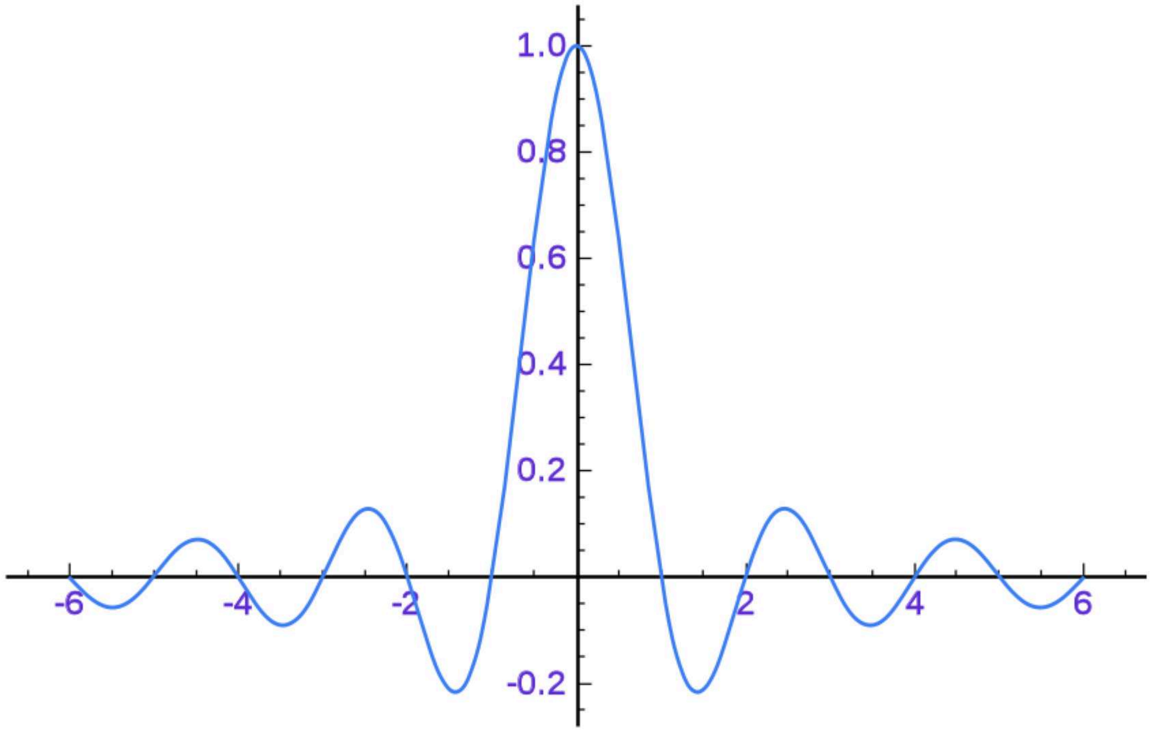

Figure 2: Standard *sinc* function, $\frac{\sin(\pi t)}{\pi t}$.

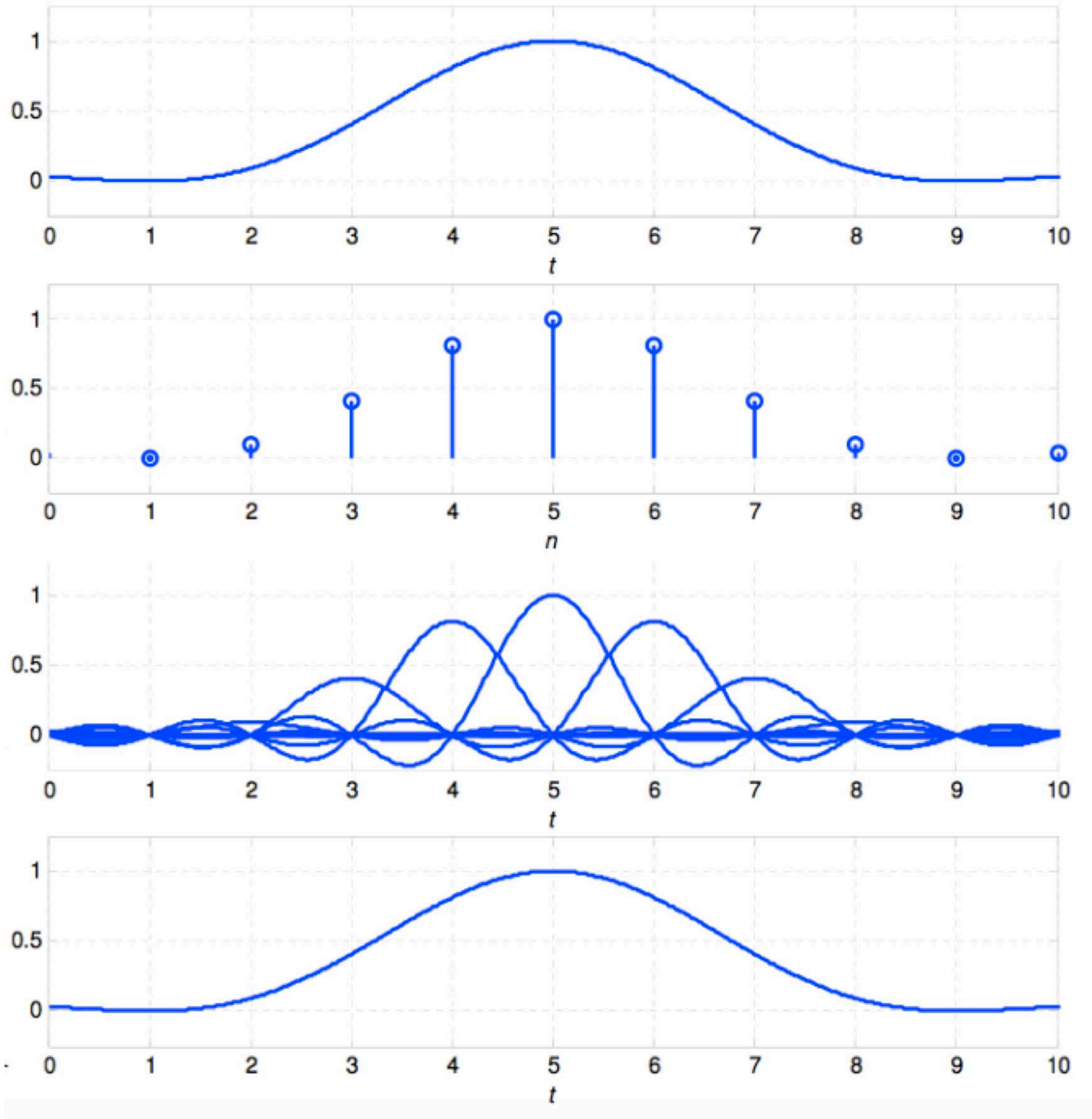

Figure 3: Perfect reconstruction of discretely sampled function by convolution. The reconstruction is a linear combination of translated *sinc* functions.

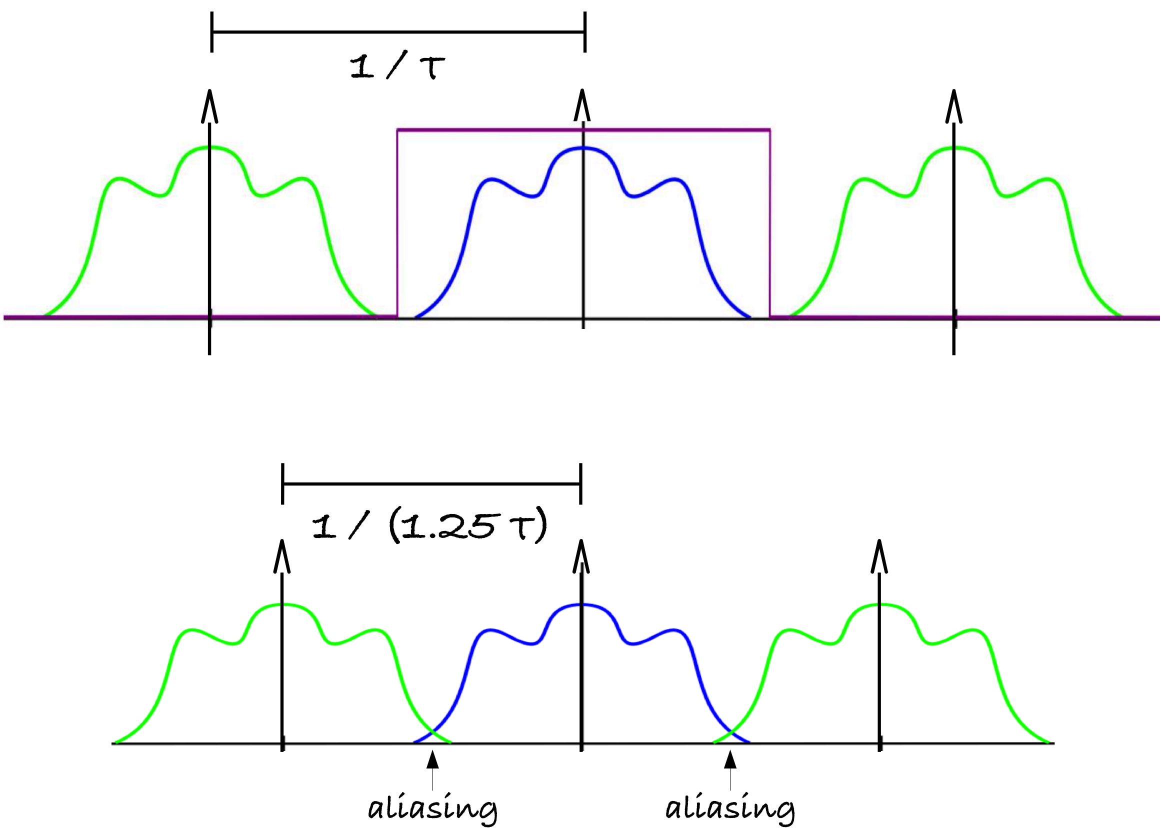

Figure 4: Original spectrum (blue) can only be recovered when it does not overlap with cloned spectra which result from downsampling (green).

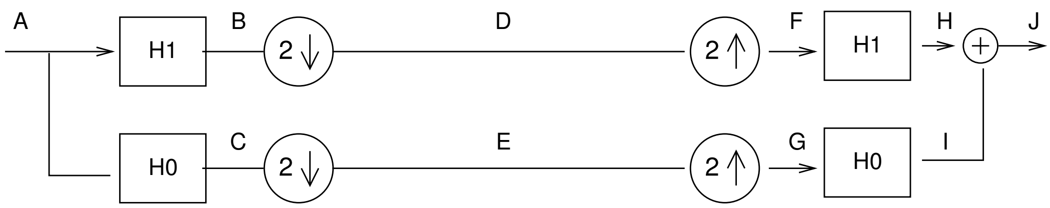

Figure 5: Two channel subband coding.

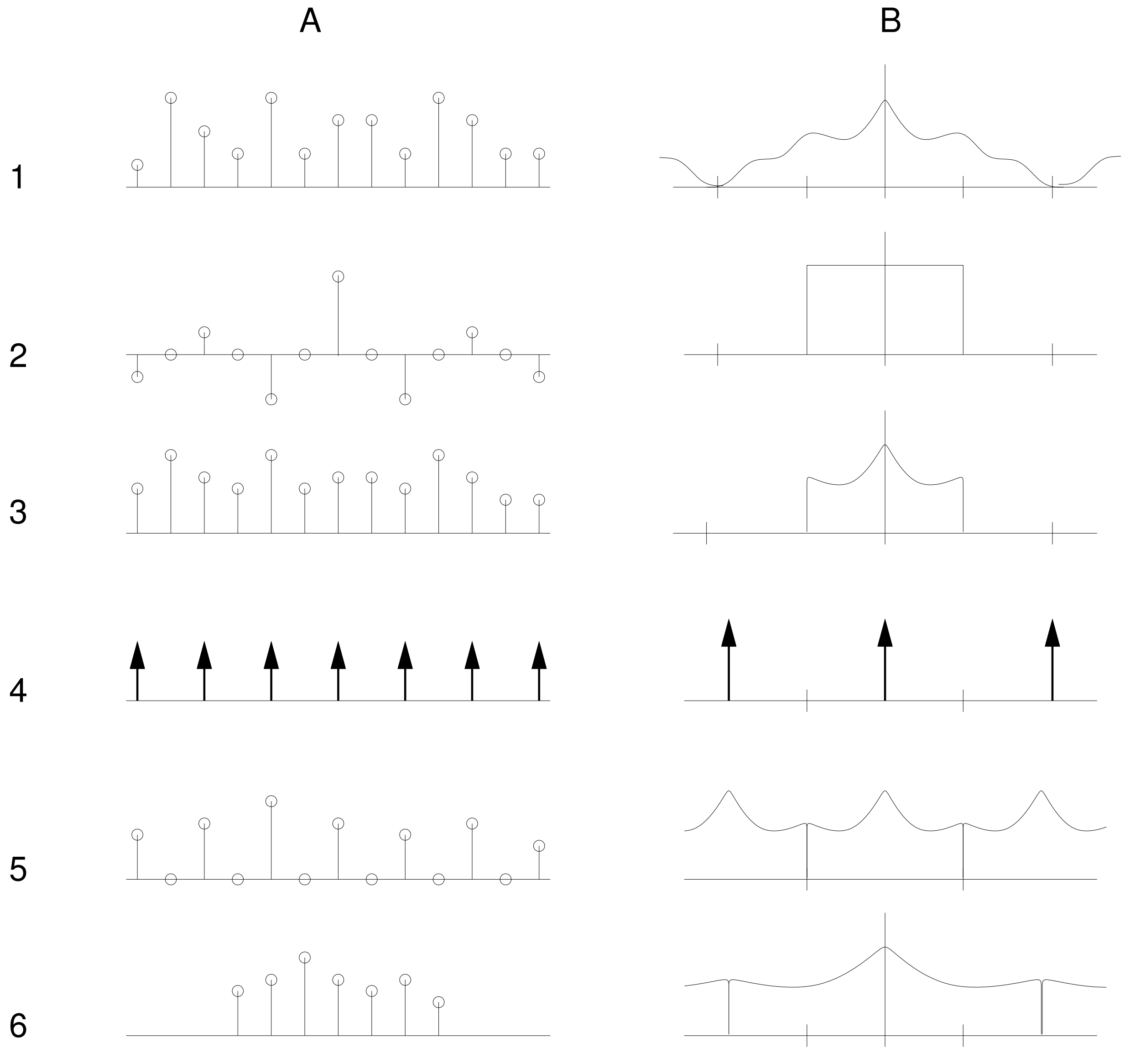

Figure 6: Lower halfband.

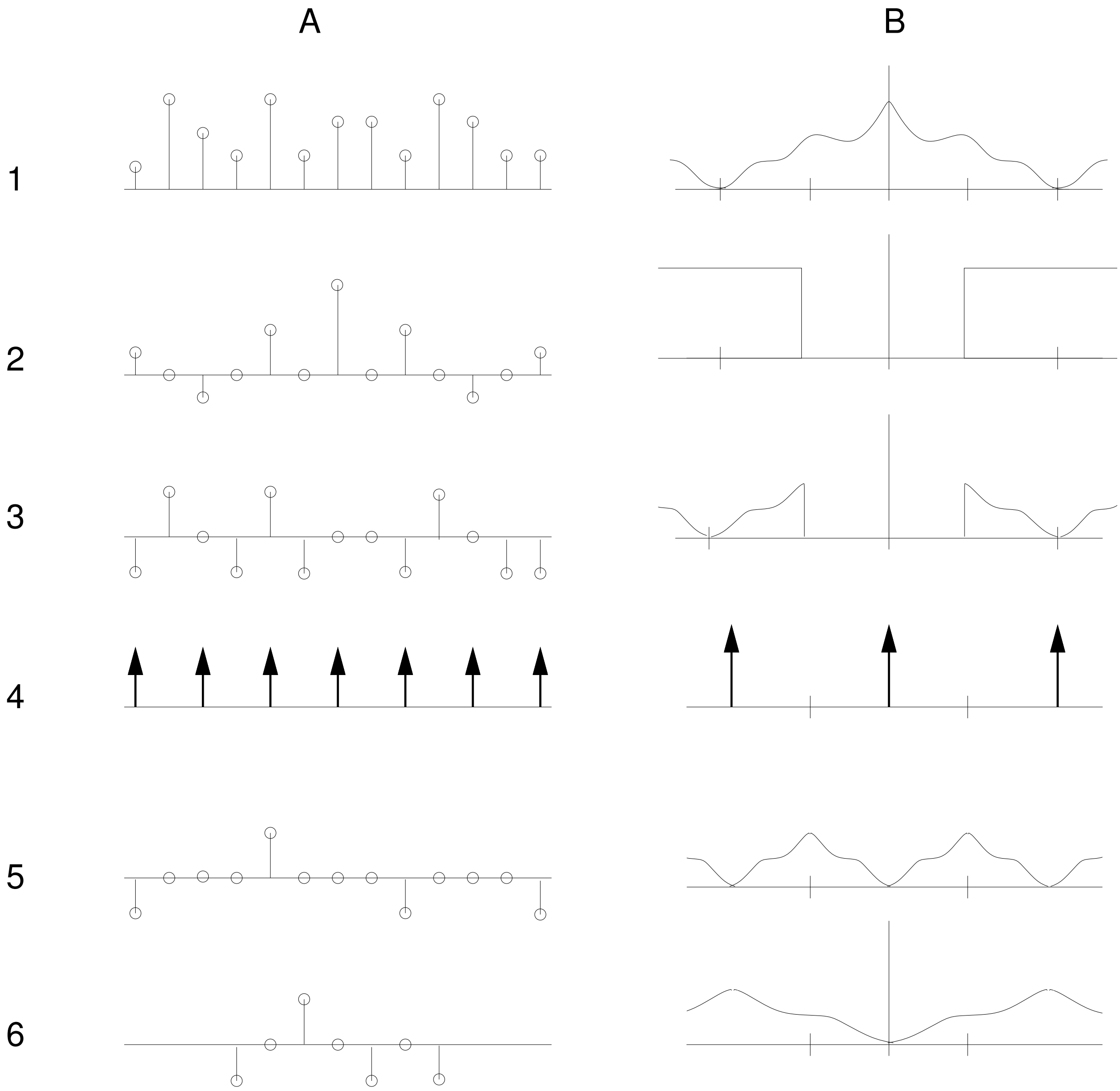

Figure 7: Upper halfband.