CS 361, Lecture 9

Jared Saia
University of New Mexico

Outline

"For NASA, space is still a high priority", Dan Quayle

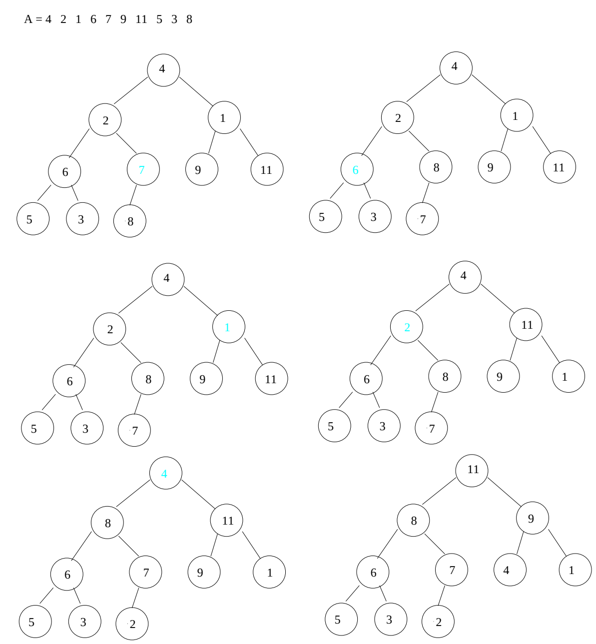
- Heap Sort
- Priority Queues
- Quicksort

Build-Max-Heap

Build-Max-Heap (A)

1. heap-size (A) = length (A)
2. for $(i = \lfloor length(A)/2 \rfloor; i > 0; i--)$
   (a) do Max-Heapify (A,i)

Example

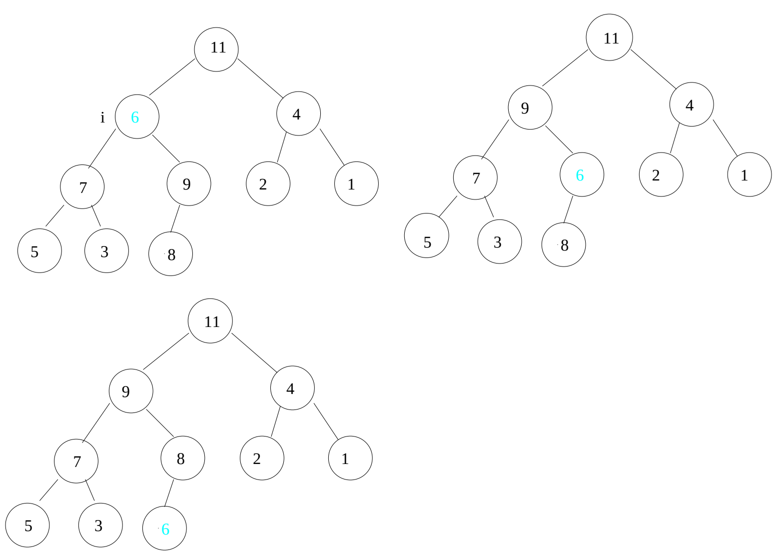


Max-Heapify

Max-Heapify (A,i)

1. $l = Left(i)$
2. $r = Right(i)$
3. $largest = i$
4. if ($l \leq$ heap-size(A) and $A[l] > A[i]$) then $largest = l$
5. if ($r \leq$ heap-size(A) and $A[r] > A[largest]$) then $largest = r$
6. if $largest \neq i$ then
   (a) exchange A[i] and A[largest]
   (b) Max-Heapify (A,largest)

Example



Heap-Sort Review

Heap-Sort (A)

1. Build-Max-Heap (A)
2. for (i=length (A);$i > 1$;$i--$)
   (a) do exchange A[1] and A[i]
   (b) heap-size (A) = heap-size (A) - 1
   (c) Max-Heapify (A,1)

Analysis

- Heap-Sort takes $O(n \log n)$ time. Q: What is best case run-time? Q: What is runtime if the array is already in sorted order?
- Q: Correctness?

Analysis

We can prove correctness by using the following loop invariant:

- At the start of each iteration of the for loop, the subarray A[1..i] is a max-heap containing the i smallest elements of A[1..n] and the subarray A[i+1..n] contains the n-i largest elements of A[1..n] in sorted order.

Priority Queues

A Priority Queue is an ADT for a set S which supports the following operations:

- *Insert (S,x)*: inserts x into the set S
- *Maximum (S)*: returns the maximum element in S
- *Extract-Max (S)*: removes and returns the element of S with the largest key
- *Increase-Key (S,x,k)*: increases the value of x's key to the new value k (k is assumed to be as large as x's current key)

(note: can also have an analagous min-priority queue)

Applications of Priority Queue

- Application: Scheduling jobs on a workstation
- Priority Queue holds jobs to be performed and their priorities
- When a job is finished or interrupted, highest-priority job is chosen using Extract-Max
- New jobs can be added using Insert

(note: an application of a min-priority queue is scheduling events in a simulation)

Implementation

- A Priority Queue can be implemented using heaps
- We'll show how to implement each of these four functions using heaps

Heap-Maximum

Heap-Maximum (A)

1. return A[1]

Heap-Extract-Max

Heap-Extract-Max (A)

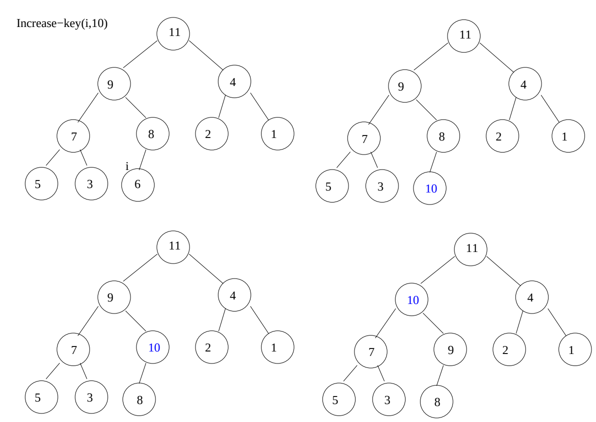
1. if (heap-size (A)<1) then return “error”
2. max = A[1];
3. A[1] = A[heap-size (A)];
4. heap-size (A)−−;
5. Max-Heapify (A,1);
6. return max;

Heap-Increase-Key

Heap-Increase-Key (A,i,key)

1. if (key < A[i]) then error “new key is smaller than current key”
2. A[i] = key;
3. while (i>1 and A[Parent (i)] < A[i])
   (a) do exchange A[i] and A[Parent (i)]
   (b) i = Parent (i);

Example



Heap-Insert

Heap-Insert (A,key)

1. heap-size (A) ++;
2. A[heap-size (A)] = - infinity
3. Heap-Increase-Key (A,heap-size (A), key)

Analysis

- Heap-Maximum takes $O(1)$ time
- Heap-Extract-Max takes $O(\log n)$
- Heap-Increase-Key takes $O(\log n)$
- Heap-Insert takes $O(\log n)$

Correctness?

In-Class Exercise

- Imagine you have a min-heap with the following operations defined and taking $O(\log n)$:
  - (key,data) Heap-Extract-Min (A)
  - Heap-Insert (A,key,data)
- Now assume you're given k sorted lists, each of length n/k
- Use this min-heap to give a $O(n \log k)$ algorithm for merging these k lists into one sorted list of size n.

In-Class Exercise

- Q1: What is the high level idea for solving this problem?
- Q2: What is the pseudocode for solving the problem?
- Q3: What is the runtime analysis?
- Q4: What would be an appropriate loop invariant for proving correctness of the algorithm?

Quicksort

- Based on divide and conquer strategy
- Worst case is $\Theta(n^2)$
- Expected running time is $\Theta(n \log n)$
- An In-place sorting algorithm
- Almost always the fastest sorting algorithm

Quicksort

- **Divide:** Pick some element A[q] of the array A and partition A into two arrays A_1 and A_2 such that every element in A_1 is $\leq$ A[q], and every element in A_2 is $>$ A[p]
- **Conquer:** Recursively sort A_1 and A_2
- **Combine:** A_1 concatenated with $A[q]$ concatenated with A_2 is now the sorted version of A

The Algorithm

```
//PRE: A is the array to be sorted, p>=1;
//     r is <= the size of A
//POST: A[p..r] is in sorted order
Quicksort (A,p,r){
  if (p<r){
    q = Partition (A,p,r);
    Quicksort (A,p,q-1);
    Quicksort (A,q+1,r);
}
```


Partition

```
//PRE: A[p..r] is the array to be partitioned, p>=1 and r <= size
//     of A, A[r] is the pivot element
//POST: Let A' be the array A after the function is run.  Then
//      A'[p..r] contains the same elements as A[p..r].  Further,
//      all elements in A'[p..res-1] are <= A[r], A'[res] = A[r],
//      and all elements in A'[res+1..r] are > A[r]
Partition (A,p,r){
  x = A[r];
  i = p-1;
  for (j=p;j<=r-1;j++){
    if (A[j]<=x){
      i++;
      exchange A[i] and A[j];
  }
  exchange A[i+1] and A[r];
  return i+1;
}
```

Correctness of Partition

Basic idea: The array is partitioned into four regions, x is the pivot

- Region 1: Region that is less than or equal to x
- Region 2: Region that is greater than x
- Region 3: Unprocessed region
- Region 4: Region that contains x only

Region 1 and 2 are growing and Region 3 is shrinking

Correctness

Basic idea: The array is partitioned into four regions, x is the pivot

- Region 1: Region that is less than or equal to x (between p and i)
- Region 2: Region that is greater than x (between $i + 1$ and $j - 1$)
- Region 3: Unprocessed region (between j and $r - 1$)
- Region 4: Region that contains x only (r)

Region 1 and 2 are growing and Region 3 is shrinking

Example

- Consider the array (2 6 4 1 5 3)

Loop Invariant

At the beginning of each iteration of the for loop, for any index k:

1. If $p \leq k \leq i$ then $A[k] \leq x$
2. If $i + 1 \leq k \leq j - 1$ then $A[k] > x$
3. If $k = r$ then $A[k] = x$

Todo

- Finish Chapter 6
- Start Chapter 7