University of Illinois at Urbana-Champaign
Department of Computer Science

# Lecture 9: June 25

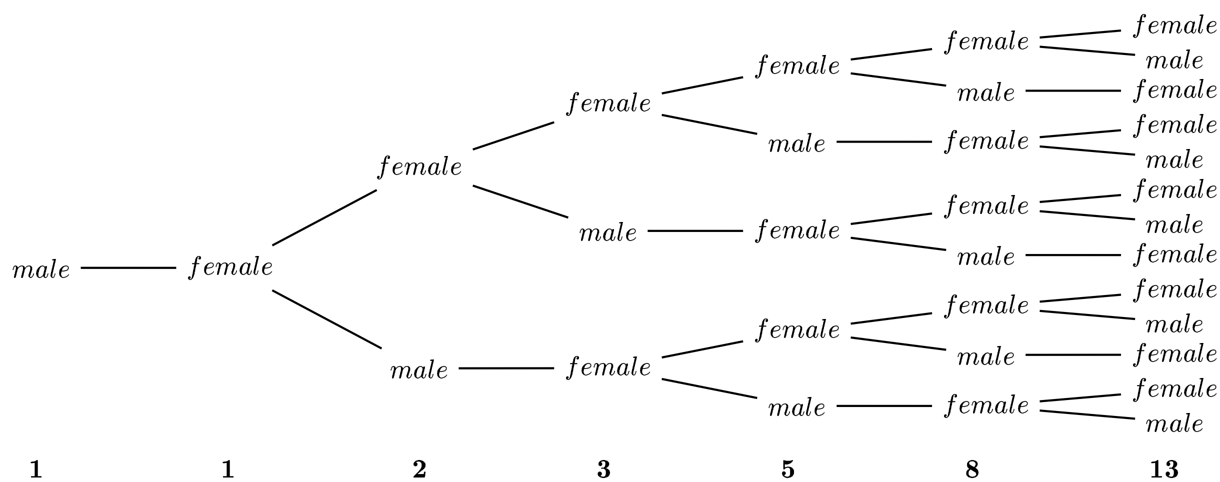
CS 273 Introduction to Theoretical Computer Science
Summer Semester, 2001

## 1 Fun with Fibonacci Numbers

> Each wife of Fibonacci,
> Eating nothing that wasn't starchy,
> Weighed as much as the two before her.
> His fifth was some signora!
> *J. A. Lindon*

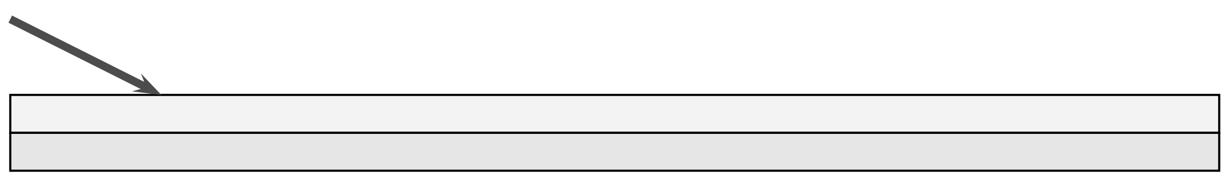
We will now shift direction towards the analysis of sequences, by using two example cases.

Consider the ancestry of a bee: a male bee (a drone) has only a female parent; a female bee has both a male and female parent. If we examine the generations we see:

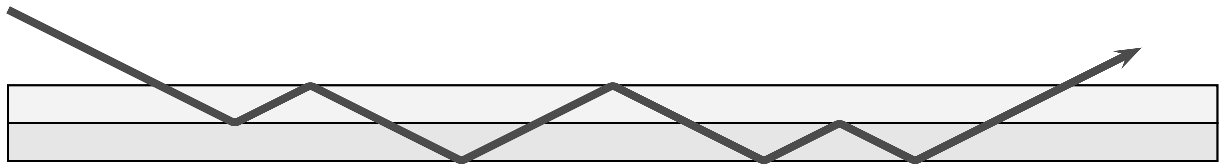



We can see that the number of ancestors in each generation is the sum of the two numbers before it. For example, our bee has 3 great-grandparents, 2 grand-parents, and 1 parent, and 3=2+1. The number of ancestors a bee has in generation $n$ is defined by the Fibonacci sequence; we can also see this by applying the rule of sum.

As a second example, consider light entering two adjacent planes of glass:

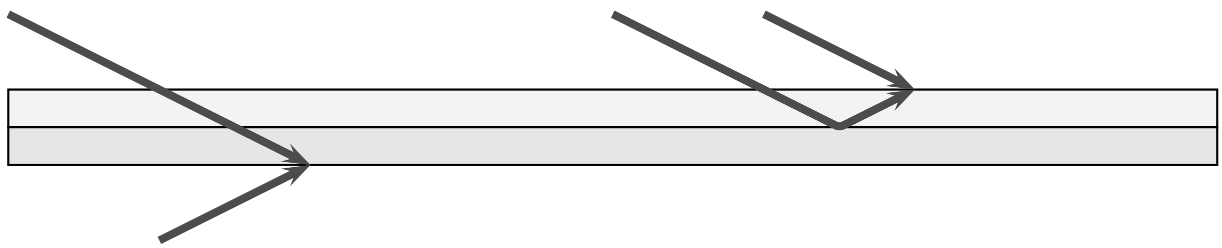


At any meeting surfaces (between the two panes of glass, or between the glass and air), the light may either reflect or continue straight through (refract) as so:



The example above shows the light bouncing 7 times before it leaves the panes. In general, how many different paths can the light take if we are told that it bounces $n$ times before leaving the glass panes?

The answer to the question (in case you haven't guessed) rests with the Fibonacci sequence. We may apply the rule of sum to the event $E$ constituting all paths through the glass in $n$ bounces. We generate two separate sub-events, $E_1$ and $E_2$; the following picture graphically illustrates aspects of these two sub-events.



**Sub-event $\mathbf{E_1}$** Let $E_1$ be the event that the first bounce **is not** through the center region between the two panes.

In this case, the light must first bounce off the bottom pane, or else we are dealing with the case of having zero bounces (there is only one way to have zero bounces). However, the number of remaining paths after bouncing off the bottom pane is the same as the number of paths entering through the bottom pane and bouncing $n-1$ bounces more. Entering through the bottom pane is the same as entering through the top pane (but flipped over), so $E_1$=the number of paths of light bouncing $n-1$ times.

**Sub-event $\mathbf{E_2}$** Let $E_2$ be the event that the first bounce **is** through the center region between the two panes.

In this case, we consider the two options for the light after the first bounce: it can either leave the glass (in which case we are dealing with the case of having one bounce through the center region, and there is only way for the light to bounce once through the center region) or it can bounce yet again on the top of the upper pane, in which case it is equivalent to the light entering from the top with $n-2$ bounces to take along its path.

By the rule of sum, we thus get the following recurrence relation for $L_n$, the number of paths in which the light can travel with $n$ bounces.

$$F_0 = 1$$

$$F_1 = 2$$

$$F_n = F_{n-1} + F_{n-2}$$

### Stump a Professor

What is a recurrence relation for three panes of glass? This question once stumped an anonymous professor in a science discipline, but now you should be able to solve it with a bit of effort. Aren't you proud of your knowledge?

# 2 Solving Recurrence Relations

"...for in laughter all that is evil comes together, but is pronounced holy and absolved by its own bliss; and if this is my alpha and omega, that all that is heavy and grave should become light; all that is body, dancer; all that is spirit bird – and verily, that is my alpha and omega: Oh how should I not lust after eternity and after the nuptial ring of rings, the ring of recurrence?"

Friedrich Nietzsche,
*Thus Spoke Zarathustra*

## 2.1 Sequence Operators and Annihilators

We have shown that several different problems can be expressed as Fibonacci sequences, but we do not know how to explicitly compute the $n$'th Fibonacci number yet. Our best method so far will have us compute the first through $n-1$'th Fibonacci number first, which requires many operations!

In order to solve recurrences such as the Fibonacci numbers, we will first understand operations on sequences of numbers. Suppose we are given a sequence of numbers $A = \langle a_0, a_1, a_2, a_3, a_4, \cdots \rangle$. We can naturally define some operators on this sequence:

- We can multiply the sequence by a constant to get:

$$cA = \langle ca_0, ca_1, ca_2, ca_3, ca_4, \cdots \rangle$$

- We can shift the sequence to the left:

$$\mathbf{E}A = \langle a_1, a_2, a_3, a_4, \cdots \rangle$$

- We can add two sequences, $A = \langle a_0, a_1, a_2, a_3, a_4, \cdots \rangle, B = \langle b_0, b_1, b_2, b_3, b_4, \cdots \rangle$ to get:

$$A + B = \langle a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, \cdots \rangle$$

We can understand these operators better by looking at an actual sequence (the powers of 2):

$$T = \langle 2^0, 2^1, 2^2, 2^3, \cdots \rangle$$

- Multiplying $T$ by a constant $c = 2$ gives:

$$2T = \langle 2 \times 2^0, 2 \times 2^1, 2 \times 2^2, 2 \times 2^3, \cdots \rangle = \langle 2^1, 2^2, 2^3, 2^4, \cdots \rangle$$

- Shifting $T$ by one place to the left, we get:

$$\mathbf{E}T = \langle 2^1, 2^2, 2^3, 2^4, \cdots \rangle$$

  where $\mathbf{E}$ is simply the left-shift operator (i.e. it shifts a sequence to the left).

- Adding the sequences $\mathbf{E}T$ and $-2T$ gives:

$$\mathbf{E}T - 2T = \langle 2^1 - 2^1, 2^2 - 2^2, 2^3 - 2^3, 2^4 - 2^4, \cdots \rangle = \langle 0, 0, 0, 0, \cdots \rangle$$

## 2.2 Properties of Operators

It turns out that the distributive property holds for these operators, so we may also rewrite $\mathbf{E}T - 2T$ as $(\mathbf{E}-2)T$. Another interesting point to realize is that $(\mathbf{E}-2)T = \langle 0,0,0,0,\cdots\rangle$. In other words, the operator $(\mathbf{E}-2)$ *annihilates* $T$ (i.e. it transforms $T$ into the sequence of zeroes); $(\mathbf{E}-2)$ is called an *annihilator* of $T$. It is obvious that multiplication by 0 will thus trivially annihilate every sequence; because of this triviality, as a technical matter, we do not consider multiplication by 0 to be an annihilator.

What if we applied the operator $(E-3)$ to our sequence $T$?:

$$\begin{aligned}(\mathbf{E}-3)T &= \mathbf{E}T + (-3)T \\ &= \langle 2^1, 2^2, 2^3, \cdots\rangle + \langle -3\times 2^0, -3\times 2^1, -3\times 2^2, \cdots\rangle \\ &= \langle (2-3)\times 2^0, (2-3)\times 2^1, (2-3)\times 2^2, \cdots\rangle \\ &= (2-3)T = -T\end{aligned}$$

The operator $(\mathbf{E}-3)$ did very little to our sequence $T$; it just flipped the sign of each number in the sequence. In fact, we will soon see that **only** $(\mathbf{E}-2)$ will annihilate $T$, and all other simple operators will affect $T$ in very minor ways. Thus, if we know the annihilator of the sequence, we also know what the sequence must look like.

In general, $(\mathbf{E}-c)$ annihilates any sequence $A$ of the form $\langle a_0 c^i\rangle$. For example, suppose you are given the recurrence $R$:

$$\begin{aligned} r_0 &= 3 \\ r_{i+1} &= 5r_i \end{aligned}$$

This recurrence is of the form $R = \langle 3\times 5^i\rangle$, as can be verified:

$$\begin{aligned} r_0 &= 3 \\ r_1 &= 5r_0 = 5\times 3 \\ r_2 &= 5r_1 = 5\times(5\times 3) = 5^2\times 3 \\ r_3 &= 5r_2 = 5\times(5^2\times 3) = 5^3\times 3 \\ &\vdots \\ r_i &= 5r_{i-1} = 5\times(5^{i-1}\times 3) = 5^i\times 3 \end{aligned}$$

Thus, $(\mathbf{E}-5)$ annihilates $R$, and you can confirm this at home by explicitly doing the transformation.

What does $(\mathbf{E}-c)$ do to other sequences $A = \langle a_0 d^i\rangle$ when $d \neq c$? Lets find out:

$$\begin{aligned}(\mathbf{E}-c)A &= (\mathbf{E}-c)\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots\rangle \\ &= \mathbf{E}\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots\rangle - c\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots\rangle \\ &= \langle a_0 d, a_0 d^2, a_0 d^3, \cdots\rangle - \langle ca_0, ca_0 d, ca_0 d^2, ca_0 d^3, \cdots\rangle \\ &= \langle a_0 d - ca_0, a_0 d^2 - ca_0 d, a_0 d^3 - ca_0 d^2, \cdots\rangle \\ &= \langle (d-c)a_0, (d-c)a_0 d, (d-c)a_0 d^2, \cdots\rangle \\ &= (d-c)\langle a_0, a_0 d, a_0 d^2, \cdots\rangle \\ &= (d-c)A\end{aligned}$$

So we have a more rigorous confirmation that an annihilator annihilates exactly one type of sequence, but multiplies other similar sequences by a constant.

## 2.3 Multiple Operators

We have studied how to apply one operator to a sequence, but what about applying more than one operator to a sequence. For example, we can multiply a sequence $A = \langle a_i \rangle$ by a constant $d$ and then by a constant $c$, resulting in the sequence $(cd)A$. Alternatively, we may multiply the sequence by a constant $c$ and then shift it to the left to get $\mathbf{E}(cT)$ which is the same thing as applying the operators in the reverse order: $c(\mathbf{E}T)$. We can also shift the sequence twice to the left, $\mathbf{E}(\mathbf{E}T)$, which we will write in shorthand as $\mathbf{E}^2T$.

We now have the tools to solve a whole host of recurrence problems. For example, what annihilates $C = \langle 2^i + 3^i \rangle$? Well, we know that $(\mathbf{E} - 2)$ annihilates $\langle 2^i \rangle$ while leaving $\langle 3^i \rangle$ essentially unscathed. Similarly, $(\mathbf{E} - 3)$ annihilates $\langle 3^i \rangle$ while leaving $\langle 2^i \rangle$ essentially unscathed. Thus, if we apply both operators, we see that $(\mathbf{E} - 2)(\mathbf{E} - 3)$ annihilates our sequence $C$.

In general, $(\mathbf{E} - a)(\mathbf{E} - b)$ will annihilate **only** all sequences of the form $\langle c_1 a^i + c_2 b^i \rangle$ (we assume $a \neq b$). We will often multiply out the operators into the shorthand notation $\mathbf{E}^2 - (a + b)\mathbf{E} + ab$. It is left as an exhilarating exercise to the student to verify that this shorthand actually does represent another way of looking at the operator (i.e. that $(\mathbf{E}^2 - (a + b)\mathbf{E} + ab)A$ gives the same sequence as $(\mathbf{E} - a)(\mathbf{E} - b)A$).