CS 461, Lecture 6

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Today's Outline

- String Alignment
- Matrix Multiplication

Example II

- Unfortunately, it can be more difficult to compute the edit distance exactly. Example:

A	L	G	O	R		I		T	H	M
A	L		T	R	U	I	S	T	I	C

Key Observation

- If we remove the last column in an optimal alignment, the remaining alignment must also be optimal
- Easy to prove by contradiction: Assume there is some better subalignment of all but the last column. Then we can just paste the last column onto this better subalignment to get a better overall alignment.
- Note: The last column can be either: 1) a blank on top aligned with a character on bottom, 2) a character on top aligned with a blank on bottom or 3) a character on top aligned with a character on bottom

DP Solution

- To develop a DP algorithm for this problem, we first need to find a recursive definition
- Assume we have a m length string A and an n length string B
- Let $E(i,j)$ be the edit distance between the first i characters of A and the first j characters of B
- Then what we want to find is $E(n,m)$

Recursive Definition

- Say we want to compute $E(i,j)$ for some i and j
- Further say that the "Recursion Fairy" can tell us the solution to $E(i',j')$, for all $i' \leq i$, $j' \leq j$, *except* for $i' = i$ and $j' = j$
- Q: Can we compute $E(i,j)$ efficiently with help from the our fairy friend?

Recursive Definition

There are three possible cases:

- **Insertion:** $E(i,j) = 1 + E(i-1,j)$
- **Deletion:** $E(i,j) = 1 + E(i,j-1)$
- **Substitution:** If $a_i = b_j$, $E(i,j) = E(i-1,j-1)$, else $E(i,j) = E(i-1,j-1) + 1$

Summary

Let $I(A[i] \neq B[j]) = 1$ if $A[i]$ and $B[j]$ are different, and 0 if they are the same. Then:

$$E(i,j) = \min \left\{ \begin{array}{l} E(i-1,j)+1, \\ E(i,j-1)+1, \\ E(i-1,j-1)+I(A[i] \neq B[j]) \end{array} \right\}$$

Base Case(s)

It's not too hard to see that:

- $E(0, j) = j$ for all j, since the j characters of B must be aligned with blanks
- Similarly, $E(i, 0) = i$ for all i

Recursive Alg

- We now have enough info to directly create a recursive algorithm
- The run time of this recursive algorithm would be given by the following recurrence:

$$T(m,0) = T(0,n) = O(1), \qquad T(m,n) = T(m,n-1) + T(m-1,n) + T$$

- $T(n,n) = \Theta(1 + \sqrt{2}^n)$, which is terribly, terribly slow.

Better Idea

- We can build up a $m \times n$ table which contains all values of $E(i, j)$
- We start by filling in the base cases for this table: the entries in the 0-th row and 0-th column
- To fill in any other entry, we need to know the values directly above, to the left and above and to the left.
- Thus we can fill in the table in the standard way: left to right and top down to ensure that the entries we need to fill in each cell are always available

		A	L	G	O	R	I	T	H	M
	0	1	2	3	4	5	6	7	8	9
A	1	**0**	1	2	3	4	5	6	7	8
L	2	1	**0**	1	2	3	4	5	6	7
T	3	2	1	1	2	3	4	**4**	5	6
R	4	3	2	2	2	**2**	3	4	5	6
U	5	4	3	3	3	3	3	4	5	6
I	6	5	4	4	4	4	**3**	4	5	6
S	7	6	5	5	5	5	4	4	5	6
T	8	7	6	6	6	6	5	**4**	5	6
I	9	8	7	7	7	7	6	5	5	6
C	10	9	8	8	8	8	7	6	6	6

The code

```
EditDistance(A[1,..,m],B[1,..,n]){
  for (i=1;i<=m;i++){
    Edit[i,0] = i;}
  for (j=1;j<=n;j++){
      Edit[0,j] = j;}
  for (i=1;i<=m;i++){
    for (j=1;j<=n;j++){
      if (A[i]==B[j]){
        Edit[i,j] = min(Edit[i-1,j]+1,
                        Edit[i,j-1]+1,
                        Edit[i-1,j-1]);
      }else{
        Edit[i,j] = min(Edit[i-1,j]+1,
                        Edit[i,j-1]+1;
                        Edit[i-1,j-1]+1);
  }}}
  return Edit[m,n];}
```


Reconstructing an optimal alignment

- In this code, we do not keep info around to reconstruct the optimal alignment
- However, it is a simple matter to keep around another array which stores, for each cell, a pointer to the cell that was used to achieve the current cell's minimum edit distance
- To reconstruct a solution, we then need only follow these pointers from the bottom right corner up to the top left corner

In Class Exercise

- Create a string alignment table for the two strings "abba" and "bab". Put "abba" at the top of the table and "bab" on the left side
- Qi: $(i = 1, 2, \ldots, 5)$ What is the i-th row of your table
- Q6: What is the minimum edit distance and how many alignments achieve it?

Take Away

- To solve the string alignment problem, we did the following: 1) formulated the problem recursively 2) built a solution to the recurrence from the bottom up
- Next we'll see how a similar technique can be used to solve the matrix multiplication problem.

Matrix Chain Multiplication

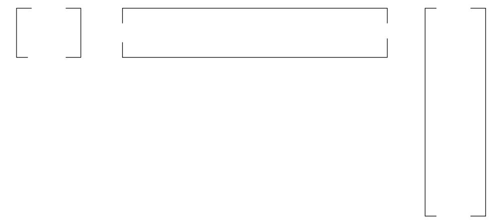
Problem:

- We are given a sequence of n matrices, $A_1, A_2, \ldots, A_n$, where for $i = 1, 2, \ldots, n$, matrix A_i has dimension p_{i-1} by p_i
- We want to compute the product, $A_1 A_2, \ldots, A_n$ as quickly as possible.
- In particular, we want to fully *paranthesize* the expression above so there are no ambiguities about the how the matrices are multiplied
- A product of matrices is *fully parenthisized* if it is either a single matrix, or the product of two fully parenthesized matrix products, sorrounded by parantheses

Paranthesizing Matrices

- There are many ways to paranthesize the matrices
- Each way gives the same output (because of associativity of matrix multiplications)
- However the way we paranthesize will effect the *time* to compute the output
- Our Goal: Find a paranthesization which requires the minimal number of scalar multiplications

Example



- In this example, it's much better to multiply the last two matrices first (this gives us a short, narrow matrix on the right)
- Worse to multiply the first two matrices first (this gives us a short wide matrix on the left)
- In general, our goal is to find ways to always create narrow and short resulting matrices.

A Problem

Problem: There can be many ways to paranthesize. E.g.

- $(A_1(A_2(A_3A_4)))$
- $(A_1((A_2A_3)A_4))$
- $((A_1A_2)(A_3A_4))$
- $((A_1(A_2A_3))A_4)$
- $(((A_1A_2)A_3)A_4)$

A Problem

- Let $P(n)$ be the number of ways to paranthesize n matrices. Then $P(1) = 1$
- For $n \geq 2$, we know that a fully paranthesized product is the product of two fully paranthesized products, and the split can occur anywhere from $k = 1$ to $k = n - 1$.
- Hence for $n \geq 2$:

$$P(n) = \sum_{k=1}^{n-1} P(k)P(n-k)$$

- In the hw, you will show that the solution to this recurrence is $\Omega(2^n)$

The Pattern

Q: Can we develop a DP Solution to this problem?

- **Formulate the problem recursively.**. Write down a formula for the whole problem as a simple combination of answers to smaller subproblems
- **Build solutions to your recurrence from the bottom up.** Write an algorithm that starts with the base cases of your recurrence and works its way up to the final solution by considering the intermediate subproblems in the correct order.

Key Observation

- Let $A_{i..j}$ (for $i \leq j$) be the matrix that results from evaluating the product $A_i A_{i+1}, \ldots A_j$
- Note that if $i < j$, then for some value of k, $i \leq k < j$, we must first compute $A_{i..k}$ and $A_{k+1..j}$, and then multiply them together to get $A_{i..j}$
- The cost of this particular parenthesization is then the cost of computing $A_{i..k}$ plus the cost of computing $A_{k+1..j}$ plus cost of multiplying $A_{i..k}$ by $A_{k+1..j}$

The Cost

- $A_{i..k}$ is a p_{i-1} by p_k matrix
- $A_{k+1..j}$ is a p_k by p_j matrix
- Thus multiplying $A_{i..k}$ and $A_{k+1..j}$ takes $p_{i-1}p_kp_j$ operations

Recursive Formulation

- Let $m(i,j)$ be the minimum cost of computing $A_{i,j}$
- We've shown that $m(i,j) \leq m(i,k) + m(k+1,j) + p_{i-1}p_kp_j$ for any $k = i, i+1, \ldots, j-1$
- Further note that the optimal parenthesization must use some value of $k = i, i+1, \ldots, j-1$. So we need only pick the best

Recursive Fromulation

- $m(i,j) = 0$ if $i = j$
- $m(i,j) = \min_{i \leq k < j}\{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\}$

The Recursive Solution

- We now have enough information to write a recursive function to solve the problem
- The recursive solution will have runtime given by the following recurrence:
- $T(1) = 1$,
- $T(n) = 1 + \sum_{k=1}^{n-1}(T(k) + T(n-k) + 1)$
- Unfortunately, the solution to this recurrence is $\Omega(2^n)$ (as shown on p. 346 of the text)

DP Solution

- Note that we must solve one subproblem for each choice of i and j satisfying $1 \leq i \leq j \leq n$
- This is only $\binom{n}{2} + n = \Theta(n^2)$ subproblems
- The recursive algorithm encounters each subproblem many times in the branches of the recursion tree.
- However, we can just compute these subproblems from the bottom up, storing the results in a table (this is the DP solution)

Example

- Consider the sequence of three matrices, A_1, A_2, A_3 whose dimensions are given by the sequence $3, 1, 2, 1, 2$
- Let's construct the tables giving the optimal parenthesization
- The (i, j) entry of the first table will give the optimal cost for computing $A_{i..j}$, the (i, j) entry of the second table will give a k value which achieves this optimal cost

Example

	1	2	3	4
1	0	6	5	10
2	-	0	2	4
3	-	-	0	4
4	-	-	-	0

Example

	1	2	3	4
1	-	1	1	1
2	-	-	2	3
3	-	-	-	3
4	-	-	-	-

Optimal Parenthesization

- Thus an optimal parenthesization is $(A_1((A_2 A_3) A_4))$