CS 362, Lecture 12

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Today's Outline

- Dynamic Tables

Pseudocode

```
Table-Insert(T,x){
  if (T.size == 0){allocate T with 1 slot;T.size=1}
  if (T.num == T.size){
    allocate newTable with 2*T.size slots;
    insert all items in T.table into newTable;
    T.table = newTable;
    T.size = 2*T.size
    }
  T.table[T.num] = x;
  T.num++
}
```


Potential Method

- Let's now analyze Table-Insert using the potential method
- Let num_i be the num value for the i-th call to Table-Insert
- Let $size_i$ be the size value for the i-th call to Table-Insert
- Then let

$$\Phi_i = 2 * num_i - size_i$$

In Class Exercise

Recall that $a_i = c_i + \Phi_i - \Phi_{i-1}$

- Show that this potential function is 0 initially and always nonnegative
- Compute a_i for the case where Table-Insert does not trigger an expansion
- Compute a_i for the case where Table-Insert does trigger an expansion (note that $num_{i-1} = num_i - 1$, $size_{i-1} = num_i - 1$, $size_i = 2 * (num_i - 1)$)

Table Delete

- We've shown that a Table-Insert has $O(1)$ amortized cost
- To implement Table-Delete, it is enough to remove (or zero out) the specified item from the table
- However it is also desirable to contract the table when the load factor gets too small
- Storage for old table can then be freed to the heap

Desirable Properties

We want to preserve two properties:

- the load factor of the dynamic table is lower bounded by some constant
- the amortized cost of a table operation is bounded above by a constant

Naive Strategy

- A natural strategy for expansion and contraction is to double table size when an item is inserted into a full table and halve the size when a deletion would cause the table to become less than half full
- This strategy guarantees that load factor of table never drops below 1/2

D'Oh

- Unfortunately this strategy can cause amortized cost of an operation to be large
- Assume we perform n operations where n is a power of 2
- The first $n/2$ operations are insertions
- At the end of this, $T.num = T.size = n/2$
- Now the remaining $n/2$ operations are as follows:

$$I, D, D, I, I, D, D, I, I, \ldots$$

where I represents an insertion and D represents a deletion

Analysis

- Note that the first insertion causes an expansion
- The two following deletions cause a contraction
- The next two insertions cause an expansion again, etc., etc.
- The cost of each expansion and deletion is $\Theta(n)$ and there are $\Theta(n)$ of them
- Thus the total cost of n operations is $\Theta(n^2)$ and so the amortized cost per operation is $\Theta(n)$

The Solution

- The Problem: After an expansion, we don't perform enough deletions to pay for the contraction (and vice versa)
- The Solution: We allow the load factor to drop below 1/2
- In particular, halve the table size when a deletion causes the table to be less than 1/4 full
- We can now create a potential function to show that Insertion and Deletion are fast in an amortized sense

Recall: Load Factor

- For a nonempty table T, we define the "load factor" of T, $\alpha(T)$, to be the number of items stored in the table divided by the size (number of slots) of the table
- We assign an empty table (one with no items) size 0 and load factor of 1
- Note that the load factor of any table is always between 0 and 1
- Further if we can say that the load factor of a table is always at least some constant c, then the unused space in the table is never more than $1 - c$

The Potential

$$\Phi(t) = \left\{ \begin{array}{ll} 2 * T.num - T.size & \text{if } \alpha(T) \geq 1/2 \\ T.size/2 - T.num & \text{if } \alpha(T) < 1/2 \end{array} \right\}$$

- Note that this potential is legal since $\Phi(0) = 0$ and (you can prove that) $\Phi(i) \geq 0$ for all i

Intuition

- Note that when $\alpha = 1/2$, the potential is 0
- When the load factor is 1 ($T.size = T.num$), $\Phi(T) = T.num$, so the potential can pay for an expansion
- When the load factor is 1/4, $T.size = 4 * T.num$, which means $\Phi(T) = T.num$, so the potential can pay for a contraction if an item is deleted

Analysis

- Let's now role up our sleeves and show that the amortized costs of insertions and deletions are small
- We'll do this by case analysis
- Let num_i be the number of items in the table after the i-th operation, $size_i$ be the size of the table after the i-th operation, and α_i denote the load factor after the i-th operation

Table Insert

- If $\alpha_{i-1} \geq 1/2$, analysis is identical to the analysis done in the In-Class Exercise - amortized cost per operation is 3
- If $\alpha_{i-1} < 1/2$, the table will not expand as a result of the operation
- There are two subcases when $\alpha_{i-1} < 1/2$: 1) $\alpha_i < 1/2$ 2) $\alpha_i \geq 1/2$

$\alpha_i < 1/2$

- In this case, we have

$$
\begin{aligned}
a_i &= c_i + \Phi_i - \Phi_{i-1} && (1)\\
&= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) && (2)\\
&= 1 + (size_i/2 - num_i) - (size_i/2 - (num_i - 1)) && (3)\\
&= 0 && (4)
\end{aligned}
$$

$\alpha_i \geq 1/2$

$$
\begin{aligned}
a_i &= c_i + \Phi_i - \Phi_{i-1} && (5)\\
&= 1 + (2 * num_i - size_i) - (size_{i-1}/2 - num_{i-1}) && (6)\\
&= 1 + (2 * (num_{i-1} + 1) - size_{i-1}) - (size_{i-1}/2 - num_{i-1}) && (7)\\
&= 3 * num_{i-1} - \frac{3}{2} size_{i-1} + 3 && (8)\\
&= 3 * \alpha_{i-1} * size_{i-1} - \frac{3}{2} size_{i-1} + 3 && (9)\\
&< \frac{3}{2} * size_{i-1} - \frac{3}{2} size_{i-1} + 3 && (10)\\
&= 3 && (11)
\end{aligned}
$$

Take Away

- So we've just show that in all cases, the amortized cost of an insertion is 3
- We did this by case analysis
- What remains to be shown is that the amortized cost of deletion is small
- We'll also do this by case analysis

Deletions

- For deletions, $num_i = num_{i-1} - 1$
- We will look at two main cases: 1) $\alpha_{i-1} < 1/2$ and 2) $\alpha_{i-1} \geq 1/2$
- For the case where $\alpha_{i-1} < 1/2$, there are two subcases: 1a) the i-th operation does not cause a contraction and 1b) the i-th operation does cause a contraction

Case 1a

- If $\alpha_{i-1} < 1/2$ and the i-th operation does not cause a contraction, we know $size_i = size_{i-1}$ and we have:

$$\begin{aligned}
a_i &= c_i + \Phi_i - \Phi_{i-1} && (12)\\
&= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) && (13)\\
&= 1 + (size_i/2 - num_i) - (size_i/2 - (num_i + 1)) && (14)\\
&= 2 && (15)
\end{aligned}$$

Case 1b

- In this case, $\alpha_{i-1} < 1/2$ and the i-th operation causes a contraction.
- We know that: $c_i = num_i + 1$
- and $size_i/2 = size_{i-1}/4 = num_{i-1} = num_i + 1$. Thus:

$$\begin{aligned}
a_i &= c_i + \Phi_i - \Phi_{i-1} && (\\
&= (num_i + 1) + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) && (\\
&= (num_i + 1) + ((num_i + 1) - num_i) - ((2num_i + 2) - (num_i + && (\\
&= 1 && (
\end{aligned}$$

Case 2

- In this case, $\alpha_{i-1} \geq 1/2$
- Proving that the amortized cost is constant for this case is left as an exercise to the diligent student
- Hint1: Q: In this case is it possible for the i-th operation to be a contraction? If so, when can this occur? Hint2: Try a case analysis on α_i.

Take Away

- Since we've shown that the amortized cost of every operation is at most a constant, we've shown that any sequence of n operations on a Dynamic table take $O(n)$ time
- Note that in our scheme, the load factor never drops below 1/4
- This means that we also never have more than 3/4 of the table that is just empty space

Disjoint Sets

- A disjoint set data structure maintains a collection $\{S_1, S_2, \ldots S_k\}$ of disjoint dynamic sets
- Each set is identified by a representative which is a member of that set
- Let's call the members of the sets *objects*.

Operations

We want to support the following operations:

- Make-Set(x): creates a new set whose only member (and representative) is x
- Union(x,y): unites the sets that contain x and y (call them S_x and S_y) into a new set that is $S_x \cup S_y$. The new set is added to the data structure while S_x and S_y are deleted. The representative of the new set is any member of the set.
- Find-Set(x): Returns a pointer to the representative of the (unique) set containing x

Analysis

- We will analyze this data structure in terms of two parameters:
  1. n, the number of Make-Set operations
  2. m, the total number of Make-Set, Union, and Find-Set operations
- Since the sets are always disjoint, each Union operation reduces the number of sets by 1
- So after $n-1$ Union operations, only one set remains
- Thus the number of Union operations is at most $n-1$

Analysis

- Note also that since the Make-Set operations are included in the total number of operations, we know that $m \geq n$
- We will in general assume that the Make-Set operations are the first n performed

- Consider a simplified version of Friendster
- Every person is an object and every set represents a social clique
- Whenever a person in the set S_1 forges a link to a person in the set S_2, then we want to create a new larger social clique $S_1 \cup S_2$ (and delete S_1 and S_2)
- We might also want to find a representative of each set, to make it easy to search through the set
- For obvious reasons, we want these operation of Union, Make-Set and Find-Set to be as fast as possible