CS 461, Lecture 17

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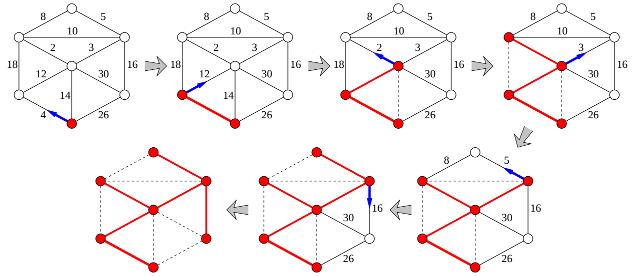
Today's Outline

- Prim's Algorithm
- Breadth First Search
- Depth First Search

Prim's Algorithm

- In Prim's algorithm, the set A maintained by the algorithm forms a single tree.
- The tree starts from an arbitrary root vertex and grows until it spans all the vertices in V
- At each step, a light edge is added to the tree A which connects A to an isolated vertex of $G_A = (V, A)$
- By our Corollary, this rule adds only safe edges to A, so when the algorithm terminates, it will return a MST

Example Run



Prim's algorithm run on the example graph, starting with the bottom vertex.
At each stage, thick edges are in A, an arrow points along A's safe edge, and dashed edges are useless.

An Implementation

- To implement Prim's algorithm, we keep all edges adjacent to A in a heap
- When we pull the minimum-weight edge off the heap, we first check to see if both its endpoints are in A
- If not, we add the edge to A and then add the neighboring edges to the heap
- If we implement Prim's algorithm this way, its running time is $O(|E| \log |E|) = O(|E| \log |V|)$
- However, we can do better

Prim's Algorithm

- We can speed things up by noticing that the algorithm visits each vertex only once
- Rather than keeping the edges in the heap, we will keep a heap of vertices, where the key of each vertex v is the weight of the minimum-weight edge between v and A (or infinity if there is no such edge)
- Each time we add a new edge to A, we may need to decrease the key of some neighboring vertices

Prim's

We will break up the algorithm into two parts, Prim-Init and Prim-Loop

```
Prim(V,E,s){
  Prim-Init(V,E,s);
  Prim-Loop(V,E,s);
}
```

Prim-Init

```
Prim-Init(V,E,s){
  for each vertex v in V - {s}{
    if ((v,s) is in E){
      edge(v) = (v,s);
      key(v) = w((v,s));
    }else{
      edge(v) = NULL;
      key(v) = infinity;
    }
  }
  Heap-Insert(v);
}
```

Prim-Loop

```
Prim-Loop(V,E,s){
  A = {};
  for (i = 1 to |V| - 1){
    v = Heap-ExtractMin();
    add edge(v) to A;
    for (each edge (u,v) in E){
      if (u is not in A AND key(u) > w(u,v)){
        edge(u) = (u,v);
        Heap-DecreaseKey(u,w(u,v));
      }
    }
  }
  return A;
}
```

Runtime?

- The runtime of Prim's is dominated by the cost of the heap operations Insert, ExtractMin and DecreaseKey
- Insert and ExtractMin are each called $O(|V|)$ times
- DecreaseKey is called $O(|E|)$ times, at most twice for each edge
- If we use a *Fibonacci Heap*, the amortized costs of Insert and DecreaseKey is $O(1)$ and the amortized cost of ExtractMin is $O(\log |V|)$
- Thus the overall run time of Prim's is $O(|E| + |V| \log |V|)$
- This is faster than Kruskal's unless $E = O(|V|)$

Note

- This analysis assumes that it is fast to find all the edges that are incident to a given vertex
- We have not yet discussed how we can do this
- This brings us to a discussion of how to represent a graph in a computer

Graph Representation

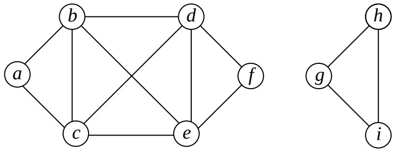
There are two common data structures used to explicity represent graphs

- Adjacency Matrices
- Adjacency Lists

Adjacency Matrix

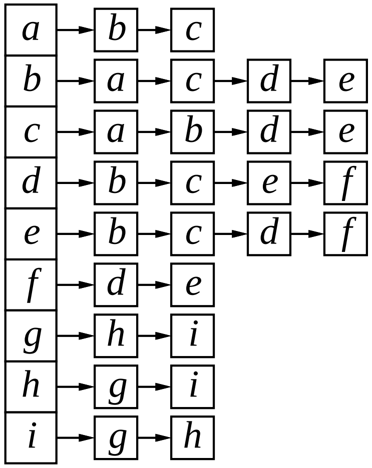
- The adjacency matrix of a graph G is a $|V| \times |V|$ matrix of 0's and 1's
- For an adjacency matrix A, the entry $A[i,j]$ is 1 if $(i,j) \in E$ and 0 otherwise
- For undirectd graphs, the adjacency matrix is always *symmetric*: $A[i,j] = A[j,i]$. Also the diagonal elements $A[i,i]$ are all zeros

Example Graph



Example Representations

	a	b	c	d	e	f	g	h	i
a	0	1	1	0	0	0	0	0	0
b	1	0	1	1	1	0	0	0	0
c	1	1	0	1	1	0	0	0	0
d	0	1	1	0	1	1	0	0	0
e	0	1	1	1	0	1	0	0	0
f	0	0	0	1	1	0	0	0	0
g	0	0	0	0	0	0	0	1	0
h	0	0	0	0	0	0	1	0	1
i	0	0	0	0	0	0	1	1	0



$a \rightarrow b \rightarrow c$

$b \rightarrow a \rightarrow c \rightarrow d \rightarrow e$

$c \rightarrow a \rightarrow b \rightarrow d \rightarrow e$

$d \rightarrow b \rightarrow c \rightarrow e \rightarrow f$

$e \rightarrow b \rightarrow c \rightarrow d \rightarrow f$

$f \rightarrow d \rightarrow e$

$g \rightarrow h \rightarrow i$

$h \rightarrow g \rightarrow i$

$i \rightarrow g \rightarrow h$

Adjacency matrix and adjacency list representations for the example graph.

Adjacency Matrix

- Given an adjacency matrix, we can decide in $\Theta(1)$ time whether two vertices are connected by an edge.
- We can also list all the neighbors of a vertex in $\Theta(|V|)$ time by scanning the row corresponding to that vertex
- This is optimal in the worst case, however if a vertex has few neighbors, we still need to examine every entry in the row to find them all
- Also, adjacency matrices require $\Theta(|V|^2)$ space, regardless of how many edges the graph has, so it is only space efficient for very *dense* graphs

Adjacency Lists

- For *sparse* graphs — graphs with relatively few edges — we're better off with adjacency lists
- An adjacency list is an array of linked lists, one list per vertex
- Each linked list stores the neighbors of the corresponding vertex

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Adjacency Lists

- The total space required for an adjacency list is $O(|V|+|E|)$
- Listing all the neighbors of a node v takes $O(1+deg(v))$ time
- We can determine if (u, v) is an edge in $O(1+deg(u))$ time by scanning the neighbor list of u
- Note that we can speed things up by storing the neighbors of a node not in lists but rather in hash tables
- Then we can determine if an edge is in the graph in expected $O(1)$ time and still list all the neighbors of a node v in $O(1+deg(v))$ time

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Take Away

- If we use the right type of heap and the right graph representation, then Prim's algorithm takes $O(|E|+|V|\log|V|)$
- This compares favorably with Kruskal's algorithm which takes $O(|E|\log|V|)$
- Kruskal's and Prims algorithms are the two main algorithms for finding the minimum spanning tree of a connected graph
- There are many, many other types of problems defined on graphs . . .

Traversing a Graph

- Suppose we want to visit every node in a connected graph (represented either explicitly or implicitly)
- The simplest way to do this is an algorithm called *depth-first search*
- We can write this algorithm recursively or iteratively - it's the same both ways, the iterative version just makes the stack explicit
- Both versions of the algorithm are initially passed a *source* vertex v

Recursive DFS

```
RecursiveDFS(v){
  if ($v$ is unmarked){
    mark $v$;
    for each edge (v,w){
      RecursiveDFS(w);
    }
  }
}
```

Iterative DFS

```
IterativeDFS(s){
  Push(s);
  while (stack not empty){
    v = Pop();
    if (v is unmarked){
      mark v;
      for each edge (v,w){
        Push(w);
      }
    }
  }
}
```

Generic Traverse

- DFS is one instance of a general family of graph traversal algorithms
- This generic graph traversal algorithm stores a set of candidate edges in a data structure we'll call a "bag"
- A "bag" is just something we can put stuff into and later take stuff out of - stacks, queues and heaps are all examples of bags.

Generic Traverse

```
Traverse(s){
  put (nil,s) in bag;
  while (the bag is not empty){
    take some edge (p,v) from the bag
    if (v is unmarked)
      mark v;
      parent(v) = p;
      for each edge (v,w){
        put (v,w) into the bag;
      }
    }
  }
}
```

Analysis

- Notice that we're keeping *edges* in the bag instead of vertices
- This is because we want to remember when we visit vertex v for teh first time, which previously-visited vertex p put v into the bag
- This vertex p is called the *parent* of v

Lemma

- Traverse(s) marks each vertex in a connected graph exactly once, and the set of edges $(v, parent(v))$, with parent(v) not nil, form a spanning tree of the graph.

Proof

- It's obvious that no node is marked more than once
- We next show that each vertex is marked at least once.
- Let $v \neq s$ be a vertex and let $s \rightarrow \cdots \rightarrow u \rightarrow v$ be the path from s to v with the minimum number of edges. (Since the graph is connected such a path always exists)
- If the algorithm marks u, then it must put (u, v) in the bag, so it must later take (u, v) out of the bag, at which point v must be marked
- Thus by induction on the shortest-path distance from s, the algorithm marks every vertex in the graph

Proof

- Call an edge $(v, parent(v))$ with $parent(v) \neq nil$ a *parent edge*
- It now remains to be shown that the parent edges form a spanning tree of the graph
- For any node v, the path of parent edges $v \rightarrow parent(v) \rightarrow parent(parent(v)) \rightarrow \cdots$ eventually leads back to s, so the set of parent edges form a connected graph.
- Since every node except s has a unique parent edge, the total number of parent edges is exactly one less than the total number of vertices
- Thus the parent edges form a spanning tree (we'll show this in the in-class exercise)

DFS and BFS

- If we implement the "bag" by using a stack, we have *Depth First Search*
- If we implement the "bag" by using a queue, we have *Breadth First Search*

Analysis

- Note that if we use adjacency lists for the graph, the overhead for the "for" loop is only a constant per edge (no matter how we implement the bag)
- If we implement the bag using either stacks or queues, each operation on the bag takes constant time
- Hence the overall runtime is $O(|V| + |E|) = O(|E|)$

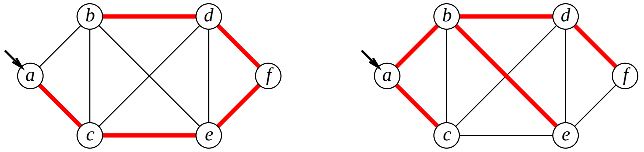
DFS vs BFS

- Note that DFS trees tend to be long and skinny while BFS trees are short and fat
- In addition, the BFS tree contains *shortest paths* from the start vertex s to every other vertex in its connected component. (here we define the length of a path to be the number of edges in the path)

Final Note

- Now assume the edges are weighted
- If we implement the "bag" using a *priority queue*, always extracting the minimum weight edge from the bag, then we have a version of Prim's algorithm
- Each extraction from the "bag" now takes $O(|E|)$ time so the total running time is $O(|V| + |E| \log |E|)$

Example



A depth-first spanning tree and a breadth-first spanning tree of one component of the example graph, with start vertex a.

In Class Exercise

- Consider a connected graph that has n vertices and $n-1$ edges. Prove by induction on n that such a graph is a tree.
- Q: What is the base case?
- Q: What is the inductive hypothesis?
- Q: What is the inductive step?