# CS 362, Lecture 2

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## Today's Outline

- L'Hopital's Rule
- Log Facts
- Recurrence Relation Review
- Recursion Tree Method
- Master Method



## L'Hopital

For any functions $f(n)$ and $g(n)$ which approach infinity and are differentiable, L'Hopital tells us that:

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$



## Example

- Q: Which grows faster $\ln n$ or $\sqrt{n}$?
- Let $f(n) = \ln n$ and $g(n) = \sqrt{n}$
- Then $f'(n) = 1/n$ and $g'(n) = (1/2)n^{-1/2}$
- So we have:

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{1/n}{(1/2)n^{-1/2}} && (1) \\ &= \lim_{n\to\infty} \frac{2}{n^{1/2}} && (2) \\ &= 0 && (3) \end{aligned}$$

- Thus $\sqrt{n}$ grows faster than $\ln n$ and so $\ln n = O(\sqrt{n})$

## A digression on logs

*It rolls down stairs alone or in pairs,*
*and over your neighbor's dog,*
*it's great for a snack or to put on your back,*
*it's log, log, log!*
*- "The Log Song" from the Ren and Stimpy Show*

- The log function shows up very frequently in algorithm analysis
- As computer scientists, when we use log, we'll mean $\log_2$ (i.e. if no base is given, assume base 2)



## Definition

- $\log_x y$ is by definition the value $z$ such that $x^z = y$
- $x^{\log_x y} = y$ by definition



## Examples

- $\log 1 = 0$
- $\log 2 = 1$
- $\log 32 = 5$
- $\log 2^k = k$

Note: $\log n$ is way, way smaller than $n$ for large values of $n$



## Examples

- $\log_3 9 = 2$
- $\log_5 125 = 3$
- $\log_4 16 = 2$
- $\log_{24} 24^{100} = 100$

## Facts about exponents

Recall that:

- $(x^y)^z = x^{yz}$
- $x^y x^z = x^{y+z}$

From these, we can derive some facts about logs



## Facts about logs

To prove both equations, raise both sides to the power of 2, and use facts about exponents

- Fact 1: $\log(xy) = \log x + \log y$
- Fact 2: $\log a^c = c \log a$

**Memorize these two facts**



## Incredibly useful fact about logs

- Fact 3: $\log_c a = \log a / \log c$

To prove this, consider the equation $a = c^{\log_c a}$, take $\log_2$ of both sides, and use Fact 2. **Memorize this fact**



## Log facts to memorize

- Fact 1: $\log(xy) = \log x + \log y$
- Fact 2: $\log a^c = c \log a$
- Fact 3: $\log_c a = \log a / \log c$

These facts are sufficient for all your logarithm needs. (You just need to figure out how to use them)

## Logs and $O$ notation

- Note that $\log_8 n = \log n / \log 8$.
- Note that $\log_{600} n^{200} = 200 * \log n / \log 600$.
- Note that $\log_{100000} 30 * n^2 = 2 * \log n / \log 100000 + \log 30 / \log 100000$
- Thus, $\log_8 n$, $\log_{600} n^{600}$, and $\log_{100000} 30 * n^2$ are all $O(\log n)$
- In general, for any constants $k_1$ and $k_2$, $\log_{k_1} n^{k_2} = k_2 \log n / \log k_1$, which is just $O(\log n)$



## Take Away

- All log functions of form $k_1 \log_{k_2} k_3 * n^{k_4}$ for constants $k_1$, $k_2$, $k_3$ and $k_4$ are $O(\log n)$
- For this reason, we don't really "care" about the base of the log function when we do asymptotic notation
- Thus, binary search, ternary search and k-ary search all take $O(\log n)$ time



## Important Note

- $\log^2 n = (\log n)^2$
- $\log^2 n$ is $O(\log^2 n)$, *not* $O(\log n)$
- This is true since $\log^2 n$ grows asymptotically faster than $\log n$
- All log functions of form $k_1 \log_{k_3}^{k_2} k_4 * n^{k_5}$ for constants $k_1$, $k_2$, $k_3$, $k_4$ and $k_5$ are $O(\log^{k_2} n)$



## In-Class Exercise

Simplify and give $O$ notation for the following functions. In the big-O notation, write all logs base 2:

- $\log 10n^2$
- $\log^2 n^4$
- $2^{\log_4 n}$
- $\log \log \sqrt{n}$

## Recurrences and Inequalities

- Often easier to prove that a recurrence is no more than some quantity than to prove that it equals something
- Consider: $f(n) = f(n-1) + f(n-2)$, $f(1) = f(2) = 1$
- "Guess" that $f(n) \leq 2^n$



## Inequalities (II)

Goal: Prove by induction that for $f(n) = f(n-1) + f(n-2)$, $f(1) = f(2) = 1$, $f(n) \leq 2^n$

- Base case: $f(1) = 1 \leq 2^1$, $f(2) = 1 \leq 2^2$
- Inductive hypothesis: for all $j < n$, $f(j) \leq 2^j$
- Inductive step:

$$\begin{aligned} f(n) &= f(n-1) + f(n-2) && (4) \\ &\leq 2^{n-1} + 2^{n-2} && (5) \\ &< 2 * 2^{n-1} && (6) \\ &= 2^n && (7) \end{aligned}$$



## Recursion-tree method

- Each node represents the cost of a single subproblem in a recursive call
- First, we sum the costs of the nodes in each level of the tree
- Then, we sum the costs of all of the levels

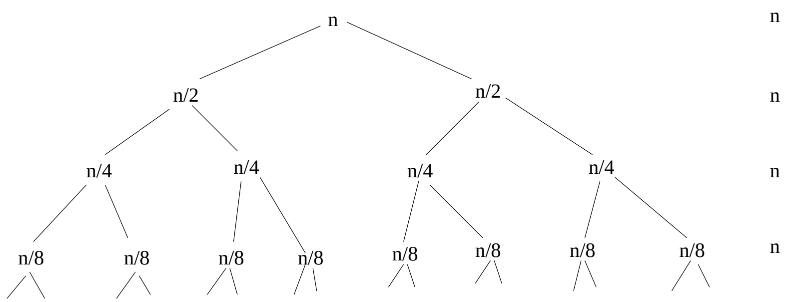


## Recursion-tree method

- Used to get a good guess which is then refined and verified using substitution method
- Best method (usually) for recurrences where a term like $T(n/c)$ appears on the right hand side of the equality

# Example 1

- Consider the recurrence for the running time of Mergesort: $T(n) = 2T(n/2) + n$, $T(1) = O(1)$





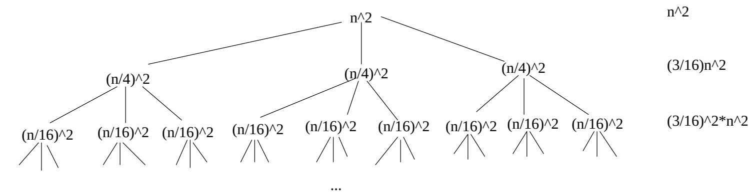

# Example 1

- We can see that each level of the tree sums to $n$
- Further the depth of the tree is $\log n$ ($n/2^d = 1$ implies that $d = \log n$).
- Thus there are $\log n + 1$ levels each of which sums to $n$
- Hence $T(n) = \Theta(n \log n)$



# Example 2

- Let's solve the recurrence $T(n) = 3T(n/4) + n^2$
- Note: For simplicity, from now on, we'll assume that $T(i) = \Theta(1)$ for all small constants $i$. This will save us from writing the base cases each time.






# Example 2

- We can see that the $i$-th level of the tree sums to $(3/16)^i n^2$.
- Further the depth of the tree is $\log_4 n$ ($n/4^d = 1$ implies that $d = \log_4 n$)
- So we can see that $T(n) = \sum_{i=0}^{\log_4 n} (3/16)^i n^2$

## Solution

$$T(n) = \sum_{i=0}^{\log_4 n} (3/16)^i n^2 \tag{8}$$
$$< n^2 \sum_{i=0}^{\infty} (3/16)^i \tag{9}$$
$$= \frac{1}{1-(3/16)} n^2 \tag{10}$$
$$= O(n^2) \tag{11}$$



## Master Theorem

- Divide and conquer algorithms often give us running-time recurrences of the form

$$T(n) = a\,T(n/b) + f(n) \tag{12}$$

- Where $a$ and $b$ are constants and $f(n)$ is some other function.
- The so-called "Master Method" gives us a general method for solving such recurrences when $f(n)$ is a simple polynomial.



## Master Theorem

- Unfortunately, the Master Theorem doesn't work for all functions $f(n)$
- Further many useful recurrences don't look like $T(n)$
- However, the theorem allows for very fast solution of recurrences when it applies



## Master Theorem

- Master Theorem is just a special case of the use of recursion trees
- Consider equation $T(n) = a\,T(n/b) + f(n)$
- We start by drawing a recursion tree

## The Recursion Tree

- The root contains the value $f(n)$
- It has $a$ children, each of which contains the value $f(n/b)$
- Each of these nodes has $a$ children, containing the value $f(n/b^2)$
- In general, level $i$ contains $a^i$ nodes with values $f(n/b^i)$
- Hence the sum of the nodes at the $i$-th level is $a^i f(n/b^i)$



## Details

- The tree stops when we get to the base case for the recurrence
- We'll assume $T(1) = f(1) = \Theta(1)$ is the base case
- Thus the depth of the tree is $\log_b n$ and there are $\log_b n + 1$ levels



## Recursion Tree

- Let $T(n)$ be the sum of all values stored in all levels of the tree:

$$T(n) = f(n) + a\,f(n/b) + a^2\,f(n/b^2) + \cdots + a^i\,f(n/b^i) + \cdots + a^L\,f(n/b^L)$$

- Where $L = \log_b n$ is the depth of the tree
- Since $f(1) = \Theta(1)$, the last term of this summation is $\Theta(a^L) = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a})$



## A "Log Fact" Aside

- It's not hard to see that $a^{\log_b n} = n^{\log_b a}$

$$\begin{aligned} a^{\log_b n} &= n^{\log_b a} && (13) \\ a^{\log_b n} &= a^{\log_a n * \log_b a} && (14) \\ \log_b n &= \log_a n * \log_b a && (15) \end{aligned}$$

- We get to the last eqn by taking $\log_a$ of both sides
- The last eqn is true by our third basic log fact

## Master Theorem

- We can now state the Master Theorem
- We will state it in a way slightly different from the book
- Note: The Master Method is just a "short cut" for the recursion tree method. It is less powerful than recursion trees.



## Master Method

The recurrence $T(n) = aT(n/b) + f(n)$ can be solved as follows:

- If $a\,f(n/b) \leq K f(n)$ for some constant $K < 1$, then $T(n) = \Theta(f(n))$.
- If $a\,f(n/b) \geq K\,f(n)$ for some constant $K > 1$, then $T(n) = \Theta(n^{\log_b a})$.
- If $a\,f(n/b) = f(n)$, then $T(n) = \Theta(f(n)\log_b n)$.



## Proof

- If $f(n)$ is a *constant factor larger* than $a\,f(n/b)$, then the sum is a descending geometric series. The sum of any geometric series is a constant times its largest term. In this case, the largest term is the first term $f(n)$.
- If $f(n)$ is a *constant factor smaller* than $a\,f(n/b)$, then the sum is an ascending geometric series. The sum of any geometric series is a constant times its largest term. In this case, this is the last term, which by our earlier argument is $\Theta(n^{\log_b a})$.
- Finally, if $a\,f(n/b) = f(n)$, then each of the $L+1$ terms in the summation is equal to $f(n)$.



## Example

- $T(n) = T(3n/4) + n$
- If we write this as $T(n) = aT(n/b) + f(n)$, then $a = 1$,$b = 4/3$,$f(n) = n$
- Here $a\,f(n/b) = 3n/4$ is smaller than $f(n) = n$ by a factor of $4/3$, so $T(n) = \Theta(n)$

## Example

- **Karatsuba's multiplication algorithm:** $\boldsymbol{T(n) = 3T(n/2) + n}$
- If we write this as $T(n) = aT(n/b) + f(n)$, then $a = 3$,$b = 2$,$f(n) = n$
- Here $a\,f(n/b) = 3n/2$ is bigger than $f(n) = n$ by a factor of $3/2$, so $T(n) = \Theta(n^{\log_2 3})$



## Example

- **Mergesort:** $\boldsymbol{T(n) = 2T(n/2) + n}$
- If we write this as $T(n) = aT(n/b) + f(n)$, then $a = 2$,$b = 2$,$f(n) = n$
- Here $a\,f(n/b) = f(n)$, so $T(n) = \Theta(n \log n)$



## Example

- $T(n) = T(n/2) + n \log n$
- If we write this as $T(n) = aT(n/b) + f(n)$, then $a = 1$,$b = 2$,$f(n) = n \log n$
- Here $a\,f(n/b) = n/2 \log n/2$ is smaller than $f(n) = n \log n$ by a constant factor, so $T(n) = \Theta(n \log n)$



## In-Class Exercise

- Consider the recurrence: $\boldsymbol{T(n) = 4T(n/2) + n \lg n}$
- Q: What is $f(n)$ and $a\,f(n/b)$?
- Q: Which of the three cases does the recurrence fall under (when $n$ is large)?
- Q: What is the solution to this recurrence?

## In-Class Exercise

- Consider the recurrence: $\boldsymbol{T(n) = 2T(n/4) + n \lg n}$
- Q: What is $f(n)$ and $a\,f(n/b)$?
- Q: Which of the three cases does the recurrence fall under (when $n$ is large)?
- Q: What is the solution to this recurrence?



## Take Away

- Recursion tree and Master method are good tools for solving many recurrences
- However these methods are limited (they can't help us get guesses for recurrences like $f(n) = f(n-1) + f(n-2)$)
- For info on how to solve these other more difficult recurrences, review the notes on annihilators on the class web page.



## Todo

- Read Chapter 3 and 4 in the text
- Work on Homework 1