CS 362, Lecture 24

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Today's Outline

- Reduction Wrapup
- Approximation algorithms for NP-Hard Problems

Hamiltonian Cycle

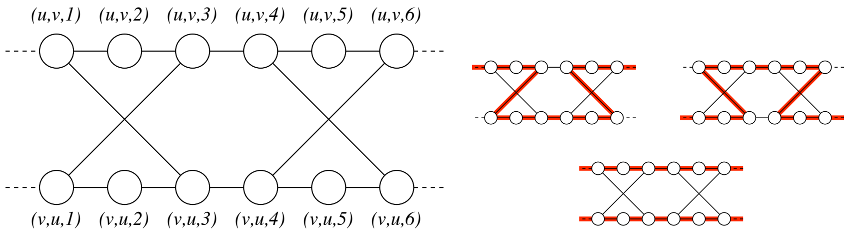
- A *Hamiltonian Cycle* in a graph is a cycle that visits every vertex exactly once (note that this is very different from an *Eulerian cycle* which visits every *edge* exactly once)
- The Hamiltonian Cycle problem is to determine if a given graph G has a Hamiltonian Cycle
- We will show that this problem is NP-Hard by a reduction from the vertex cover problem.

The Reduction

- To do the reduction, we need to show that we can solve Vertex Cover in polynomial time if we have a polynomial time solution to Hamiltonian Cycle.
- Given a graph G and an integer k, we will create another graph G' such that G' has a Hamiltonian cycle iff G has a vertex cover of size k
- As for the last reduction, our transformation will consist of putting together several "gadgets"

Edge Gadget and Cover Vertices

- For each edge (u, v) in G, we have an *edge gadget* in G' consisting of twelve vertices and fourteen edges, as shown below



An edge gadget for (u, v) and the only possible Hamiltonian paths through it.

Edge Gadget

- The four corner vertices $(u, v, 1)$, $(u, v, 6)$, $(v, u, 1)$, and $(v, u, 6)$ each have an edge leaving the gadget
- A Hamiltonian cycle can only pass through an edge gadget in one of the three ways shown in the figure
- These paths through the edge gadget will correspond to one or both of the vertices u and v being in the vertex cover.

Cover Vertices

- G' also contains k *cover vertices*, simply numbered 1 through k

Vertex Chains

- For each vertex u in G, we string together all the edge gadgets for edges (u, v) into a single *vertex chain* and then connect the ends of the chain to all the cover vertices
- Specifically, suppose u has d neighbors $v_1, v_2, \ldots, v_d$. Then G' has the following edges:
  - $d - 1$ edges between $(u, v_i, 6)$ and $(u, v_{i+1}, 1)$ (for all i between 1 and $d - 1$)
  - k edges between the cover vertices and $(u, v_1, 1)$
  - k edges between the cover vertices and $(u, v_d, 6)$

The Reduction

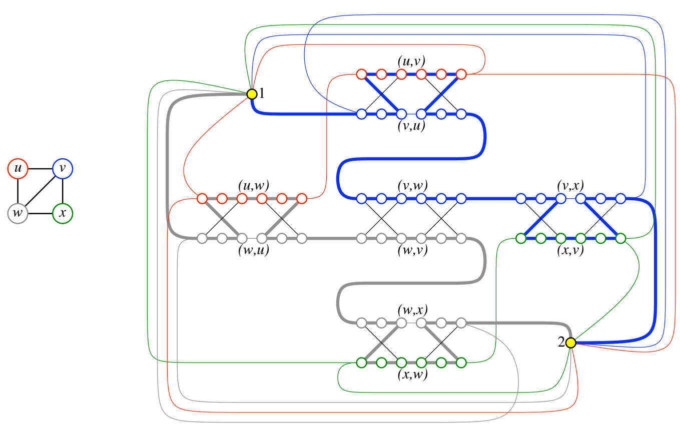
- It's not hard to prove that if $\{v_1, v_2, \ldots, v_k\}$ is a vertex cover of G, then G' has a Hamiltonian cycle
- To get this Hamiltonian cycle, we start at cover vertex 1, traverse through the vertex chain for v_1, then visit cover vertex 2, then traverse the vertex chain for v_2 and so forth, until we eventually return to cover vertex 1
- Conversely, one can prove that any Hamiltonian cycle in G' alternates between cover vertices and vertex chains, and that the vertex chains correspond to the k vertices in a vertex cover of G

Thus, G has a vertex cover of size k iff G' has a Hamiltonian cycle

The Reduction

- The transformation from G to G' takes at most $O(|V|^2)$ time, so the Hamiltonian cycle problem is NP-Hard
- Moreover we can easily verify a Hamiltonian cycle in linear time, thus Hamiltonian cycle is also in NP
- Thus Hamiltonian Cycle is NP-Complete

Example



The original graph G with vertex cover $\{v, w\}$, and the transformed graph G' with a corresponding Hamiltonian cycle (bold edges).
Vertex chains are colored to match their corresponding vertices.

The Reduction

$$\begin{tikzcd}
\boxed{\text{graph } G=(V,E),\ k} \arrow[r, "O(|V|^2)"] & \boxed{\text{graph } G'} \arrow[d, Rightarrow, "\text{Hamiltonian Cycle}"] \\
\boxed{\text{True or False}} & \boxed{\text{True or False}} \arrow[l, "O(1)"']
\end{tikzcd}$$

Traveling Sales Person

- A problem closely related to Hamiltonian cycle is the famous *Traveling Salesperson Problem(TSP)*
- The TSP problem is: "Given a weighted graph G, find the shortest cycle that visits every vertex.
- Finding the shortest cycle is obviously harder than determining if a cycle exists at all, so since Hamiltonian Cycle is NP-hard, TSP is also NP-hard!

NP-Hard Games

- In 1999, Richard Kaye proved that the solitaire game Minesweeper is NP-Hard, using a reduction from Circuit Satifiability.
- Also in the last few years, Eric Demaine, et. al., proved that the game Tetris is NP-Hard

Challenge Problem

- Consider the *optimization* version of, say, the graph coloring problem: "Given a graph G, what is the smallest number of colors needed to color the graph?" (Note that unlike the *decision* version of this problem, this is not a yes/no question)
- Show that the optimization version of graph coloring is also NP-Hard by a reduction from the decision version of graph coloring.
- Is the optimization version of graph coloring also NP-Complete?

Challenge Problem

- Consider the problem 4Sat which is: "Is there any assignment of variables to a 4CNF formula that makes the formula evaluate to true?"
- Is this problem NP-Hard? If so, give a reduction from 3Sat that shows this. If not, give a polynomial time algorithm which solves it.

Challenge Problem

- Consider the following problem: "Does there exist a clique of size 5 in some input graph G?"
- Is this problem NP-Hard? If so, prove it by giving a reduction from some known NP-Hard problem. If not, give a polynomial time algorithm which solves it.

Vertex Cover

- A *vertex cover* of a graph is a set of vertices that touches every edge in the graph
- The decision version of *Vertex Cover* is: "Does there exist a vertex cover of size k in a graph G?".
- We've proven this problem is NP-Hard by an easy reduction from Independent Set
- The *optimization* version of *Vertex Cover* is: "What is the minimum size vertex cover of a graph G?"
- We can prove this problem is NP-Hard by a reduction from the decision version of Vertex Cover (left as an exercise).

Approximating Vertex Cover

- Even though the optimization version of Vertex Cover is NP-Hard, it's possible to *approximate* the answer efficiently
- In particular, in polynomial time, we can find a vertex cover which is no more than 2 times as large as the minimal vertex cover

Approximation Algorithm

- The approximation algorithm does the following until G has no more edges:
- It chooses an arbitrary edge (u, v) in G and includes both u and v in the cover
- It then removes from G all edges which are incident to either u or v

Approximation Algorithm

```
Approx-Vertex-Cover(G){
  C = {};
  E' = Edges of G;
  while(E' is not empty){
    let (u,v) be an arbitrary edge in E';
    add both u and v to C;
    remove from E' every edge incident to u or v;
  }
  return C;
}
```


Analysis

- If we implement the graph with adjacency lists, each edge need be touched at most once
- Hence the run time of the algorithm will be $O(|V| + |E|)$, which is polynomial time
- First, note that this algorithm does in fact return a vertex cover since it ensures that every edge in G is incident to some vertex in C
- Q: Is the vertex cover actually no more than twice the optimal size?

Analysis

- Let A be the set of edges which are chosen in the first line of the while loop
- Note that no two edges of A share an endpoint
- Thus, *any* vertex cover must contain at least one endpoint of each edge in A
- Thus if $C*$ is an optimal cover then we can say that $|C*| \geq |A|$
- Further, we know that $|C| = 2|A|$
- This implies that $|C| \leq 2|C*|$

Which means that the vertex cover found by the algorithm is no more than twice the size of an optimal vertex cover.

TSP

- An optimization version of the TSP problem is: "Given a weighted graph G, what is the shortest Hamiltonian Cycle of G?"
- This problem is NP-Hard by a reduction from Hamiltonian Cycle
- However, there is a 2-approximation algorithm for this problem if the edge weights obey the *triangle inequality*

Triangle Inequality

- In many practical problems, it's reasonable to make the assumption that the weights, c, of the edges obey the *triangle inequality*
- The triangle inequality says that for all vertices $u, v, w \in V$:

$$c(u, w) \leq c(u, v) + c(v, w)$$

- In other words, the cheapest way to get from u to w is always to just take the edge (u, w)
- In the real world, this is usually a pretty natural assumption. For example it holds if the vertices are points in a plane and the cost of traveling between two vertices is just the euclidean distance between them.

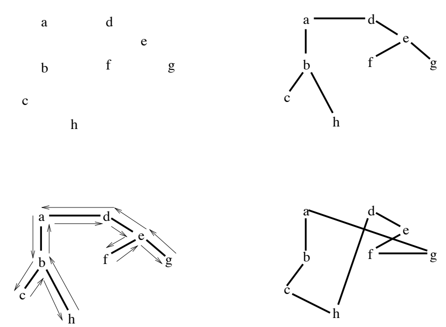
Approximation Algorithm

- Given a weighted graph G, the algorithm first computes a MST for G, T, and then arbitrarily selects a root node r of T.
- It then lets L be the list of the vertices visited in a depth first traversal of T starting at r.
- Finally, it returns the Hamiltonian Cycle, H, that visits the vertices in the order L.

Approximation Algorithm

```
Approx-TSP(G){
  T = MST(G);
  L = the list of vertices visited in a depth first traversal
      of T, starting at some arbitrary node in T;
  H = the Hamiltonian Cycle that visits the vertices in the
      order L;
  return H;
}
```


Example Run



The top left figure shows the graph G (edge weights are just the Euclidean distances between vertices); the top right figure shows the MST T. The bottom left figure shows the depth first walk on T, $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$; the bottom right figure shows the Hamiltonian cycle H obtained by deleting repeat visits from W, $H = (a, b, c, h, d, e, f, g)$.

Analysis

- The first step of the algorithm takes $O(|E| + |V| \log |V|)$ (if we use Prim's algorithm)
- The second step is $O(|V|)$
- The third step is $O(|V|)$.
- Hence the run time of the entire algorithm is polynomial

Analysis

An important fact about this algorithm is that: *the cost of the MST is less than the cost of the shortest Hamiltonian cycle.*

- To see this, let T be the MST and let $H*$ be the shortest Hamiltonian cycle.
- Note that if we remove one edge from $H*$, we have a spanning tree, T'
- Finally, note that $w(H*) \geq w(T') \geq w(T)$
- Hence $w(H*) \geq w(T)$

Analysis

- Now let W be a depth first walk of T which traverses each edge exactly twice (similar to what you did in the hw)
- In our example, $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$
- Note that $c(W) = 2c(T)$
- This implies that $c(W) \leq 2c(H*)$

Analysis

- Unfortunately, W is not a Hamiltonian cycle since it visits some vertices more than once
- However, we can delete a visit to any vertex and the cost will not increase *because of the triangle inequality*. (The path without an intermediate vertex can only be shorter)
- By repeatedly applying this operation, we can remove from W all but the first visit to each vertex, without increasing the cost of W.
- In our example, this will give us the ordering $H = (a, b, c, h, d, e, f, g)$

Analysis

- By the last slide, $c(H) \leq c(W)$.
- So $c(H) \leq c(W) = 2c(T) \leq 2c(H*)$
- Thus, $c(H) \leq 2c(H*)$
- In other words, the Hamiltonian cycle found by the algorithm has cost no more than twice the shortest Hamiltonian cycle.

Take Away

- Many real-world problems can be shown to not have an efficient solution unless $P = NP$ (these are the NP-Hard problems)
- However, if a problem is shown to be NP-Hard, all hope is not lost!
- In many cases, we can come up with an provably good approximation algorithm for the NP-Hard problem.