CS 362, Lecture 3

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Today's Outline

"Listen and Understand! That terminator is out there. It can't be bargained with, it can't be reasoned with! It doesn't feel pity, remorse, or fear. And it absolutely will not stop, ever, until you are dead!" - The Terminator

- Solving Recurrences using Annihilators

Intro to Annihilators

- Suppose we are given a sequence of numbers $A = \langle a_0, a_1, a_2, \cdots \rangle$
- This might be a sequence like the Fibonacci numbers
- I.e. $A = \langle a_0, a_1, a_2, \ldots) = (T(1), T(2), T(3), \cdots \rangle$

Annihilator Operators

We define three basic operations we can perform on this sequence:

1. Multiply the sequence by a constant: $cA = \langle ca_0, ca_1, ca_2, \cdots \rangle$
2. Shift the sequence to the left: $\mathbf{L}A = \langle a_1, a_2, a_3, \cdots \rangle$
3. Add two sequences: if $A = \langle a_0, a_1, a_2, \cdots \rangle$ and $B = \langle b_0, b_1, b_2, \cdots \rangle$, then $A + B = \langle a_0 + b_0, a_1 + b_1, a_2 + b_2, \cdots \rangle$

Annihilator Description

- We first express our recurrence as a sequence T
- *We use these three operators to "annihilate"* T*, i.e. make it all 0's*
- Key rule: can't multiply by the constant 0
- We can then determine the solution to the recurrence from the sequence of operations performed to annihilate T

Example

- Consider the recurrence $T(n) = 2T(n-1)$, $T(0) = 1$
- If we solve for the first few terms of this sequence, we can see they are $\langle 2^0, 2^1, 2^2, 2^3, \cdots \rangle$
- Thus this recurrence becomes the sequence:

$$T = \langle 2^0, 2^1, 2^2, 2^3, \cdots \rangle$$

Example (II)

Let's annihilate $T = \langle 2^0, 2^1, 2^2, 2^3, \cdots \rangle$

- Multiplying by a constant $c = 2$ gets:

$$2T = \langle 2 * 2^0, 2 * 2^1, 2 * 2^2, 2 * 2^3, \cdots \rangle = \langle 2^1, 2^2, 2^3, 2^4, \cdots \rangle$$

- Shifting one place to the left gets $\mathbf{L}T = \langle 2^1, 2^2, 2^3, 2^4, \cdots \rangle$
- Adding the sequence $\mathbf{L}T$ and $-2T$ gives:

$$\mathbf{L}T - 2T = \langle 2^1 - 2^1, 2^2 - 2^2, 2^3 - 2^3, \cdots \rangle = \langle 0, 0, 0, \cdots \rangle$$

- The annihilator of T is thus $\mathbf{L} - 2$

Distributive Property

- The distributive property holds for these three operators
- Thus can rewrite $\mathbf{L}T - 2T$ as $(\mathbf{L} - 2)T$
- The operator $(\mathbf{L} - 2)$ annihilates T (makes it the sequence of all 0's)
- Thus $(\mathbf{L} - 2)$ is called the *annihilator* of T

0, the "Forbidden Annihilator"

- Multiplication by 0 will annihilate *any* sequence
- Thus we disallow multiplication by 0 as an operation
- In particular, we disallow $(c-c) = 0$ for any c as an annihilator
- Must always have at least one **L** operator in any annihilator!

Uniqueness

- An annihilator annihilates exactly *one* type of sequence
- In general, the annihilator $\mathbf{L} - c$ annihilates any sequence of the form $\langle a_0 c^n \rangle$
- If we find the annihilator, we can find the type of sequence, and thus solve the recurrence
- We will need to use the base case for the recurrence to solve for the constant a_0

Example

If we apply operator $(\mathbf{L} - 3)$ to sequence T above, it fails to annihilate T

$$\begin{aligned}
(\mathbf{L} - 3)T &= \mathbf{L}T + (-3)T \\
&= \langle 2^1, 2^2, 2^3, \cdots \rangle + \langle -3 \times 2^0, -3 \times 2^1, -3 \times 2^2, \cdots \rangle \\
&= \langle (2-3) \times 2^0, (2-3) \times 2^1, (2-3) \times 2^2, \cdots \rangle \\
&= (2-3)T = -T
\end{aligned}$$

Example (II)

What does $(\mathbf{L} - c)$ do to other sequences $A = \langle a_0 d^n \rangle$ when $d \neq c$?:

$$\begin{aligned}
(\mathbf{L} - c)A &= (\mathbf{L} - c)\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle \\
&= \mathbf{L}\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle - c\langle a_0, a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle \\
&= \langle a_0 d, a_0 d^2, a_0 d^3, \cdots \rangle - \langle c a_0, c a_0 d, c a_0 d^2, c a_0 d^3, \cdots \rangle \\
&= \langle a_0 d - c a_0, a_0 d^2 - c a_0 d, a_0 d^3 - c a_0 d^2, \cdots \rangle \\
&= \langle (d-c) a_0, (d-c) a_0 d, (d-c) a_0 d^2, \cdots \rangle \\
&= (d-c)\langle a_0, a_0 d, a_0 d^2, \cdots \rangle \\
&= (d-c)A
\end{aligned}$$

Uniqueness

- The last example implies that an annihilator annihilates one type of sequence, but does not annihilate other types of sequences
- Thus Annihilators can help us classify sequences, and thereby solve recurrences

Lookup Table

- The annihilator $\mathbf{L} - a$ annihilates any sequence of the form $\langle c_1 a^n \rangle$

Example

First calculate the annihilator:

- Recurrence: $T(n) = 4 * T(n-1)$, $T(0) = 2$
- Sequence: $T = \langle 2, 2*4, 2*4^2, 2*4^3, \cdots \rangle$
- Calulate the annihilator:
  - $\mathbf{L}T = \langle 2*4, 2*4^2, 2*4^3, 2*4^4, \cdots \rangle$
  - $4T = \langle 2*4, 2*4^2, 2*4^3, 2*4^4, \cdots \rangle$
  - Thus $\mathbf{L}T - 4T = \langle 0, 0, 0, \cdots \rangle$
  - And so $\mathbf{L} - 4$ is the annihilator

Example (II)

Now use the annihilator to solve the recurrence

- Look up the annihilator in the "Lookup Table"
- It says: "The annihilator $\mathbf{L} - 4$ annihilates any sequence of the form $\langle c_1 4^n \rangle$"
- Thus $T(n) = c_1 4^n$, but what is c_1?
- We know $T(0) = 2$, so $T(0) = c_1 4^0 = 2$ and so $c_1 = 2$
- Thus $T(n) = 2 * 4^n$

In Class Exercise

Consider the recurrence $T(n) = 3 * T(n-1)$, $T(0) = 3$,

- Q1: Calculate $T(0)$,$T(1)$,$T(2)$ and $T(3)$ and write out the sequence T
- Q2: Calculate $\mathbf{L}T$, and use it to compute the annihilator of T
- Q3: Look up this annihilator in the lookup table to get the general solution of the recurrence for $T(n)$
- Q4: Now use the base case $T(0) = 3$ to solve for the constants in the general solution

Multiple Operators

- We can apply multiple operators to a sequence
- For example, we can multiply by the constant c and then by the constant d to get the operator cd
- We can also multiply by c and then shift left to get $c\mathbf{L}T$ which is the same as $\mathbf{L}cT$
- We can also shift the sequence twice to the left to get $\mathbf{LL}T$ which we'll write in shorthand as $\mathbf{L}^2T$

Multiple Operators

- We can string operators together to annihilate more complicated sequences
- Consider: $T = \langle 2^0 + 3^0, 2^1 + 3^1, 2^2 + 3^2, \cdots \rangle$
- We know that $(\mathbf{L}-2)$ annihilates the powers of 2 while leaving the powers of 3 essentially untouched
- Similarly, $(\mathbf{L}-3)$ annihilates the powers of 3 while leaving the powers of 2 essentially untouched
- Thus if we apply both operators, we'll see that $(\mathbf{L}-2)(\mathbf{L}-3)$ annihilates the sequence T

The Details

- Consider: $T = \langle a^0 + b^0, a^1 + b^1, a^2 + b^2, \cdots \rangle$
- $\mathbf{L}T = \langle a^1 + b^1, a^2 + b^2, a^3 + b^3, \cdots \rangle$
- $aT = \langle a^1 + a * b^0, a^2 + a * b^1, a^3 + a * b^2, \cdots \rangle$
- $\mathbf{L}T - aT = \langle (b-a)b^0, (b-a)b^1, (b-a)b^2, \cdots \rangle$
- We know that $(\mathbf{L}-a)T$ annihilates the a terms and multiplies the b terms by $b-a$ (a constant)
- Thus $(\mathbf{L}-a)T = \langle (b-a)b^0, (b-a)b^1, (b-a)b^2, \cdots \rangle$
- And so the sequence $(\mathbf{L}-a)T$ is annihilated by $(\mathbf{L}-b)$
- Thus the annihilator of T is $(\mathbf{L}-b)(\mathbf{L}-a)$

Key Point

- In general, the annihilator $(\mathbf{L}-a)(\mathbf{L}-b)$ (where $a \neq b$) will anihilate *only* all sequences of the form $\langle c_1 a^n + c_2 b^n \rangle$
- We will often multiply out $(\mathbf{L}-a)(\mathbf{L}-b)$ to $\mathbf{L}^2-(a+b)\mathbf{L}+ab$
- Left as an exercise to show that $(\mathbf{L}-a)(\mathbf{L}-b)T$ is the same as $(\mathbf{L}^2-(a+b)\mathbf{L}+ab)T$

Lookup Table

- The annihilator $\mathbf{L}-a$ annihilates sequences of the form $\langle c_1 a^n \rangle$
- The annihilator $(\mathbf{L}-a)(\mathbf{L}-b)$ (where $a \neq b$) anihilates sequences of the form $\langle c_1 a^n + c_2 b^n \rangle$

Fibonnaci Sequence

- We now know enough to solve the Fibonnaci sequence
- Recall the Fibonnaci recurrence is $T(0)=0$, $T(1)=1$, and $T(n)=T(n-1)+T(n-2)$
- Let T_n be the n-th element in the sequence
- Then we've got:

$$T = \langle T_0, T_1, T_2, T_3, \cdots \rangle \tag{1}$$
$$\mathbf{L}T = \langle T_1, T_2, T_3, T_4, \cdots \rangle \tag{2}$$
$$\mathbf{L}^2T = \langle T_2, T_3, T_4, T_5, \cdots \rangle \tag{3}$$

- Thus $\mathbf{L}^2T - \mathbf{L}T - T = \langle 0,0,0,\cdots \rangle$
- In other words, $\mathbf{L}^2-\mathbf{L}-1$ is an annihilator for T

Factoring

- $\mathbf{L}^2-\mathbf{L}-1$ is an annihilator that is not in our lookup table
- However, we can *factor* this annihilator (using the quadratic formula) to get something similar to what's in the lookup table
- $\mathbf{L}^2-\mathbf{L}-1=(\mathbf{L}-\phi)(\mathbf{L}-\hat{\phi})$, where $\phi=\frac{1+\sqrt{5}}{2}$ and $\hat{\phi}=\frac{1-\sqrt{5}}{2}$.

Quadratic Formula

"Me fail English? That's Unpossible!" - Ralph, the Simpsons

High School Algebra Review:

- To factor something of the form $ax^2 + bx + c$, we use the *Quadratic Formula*:
- $ax^2 + bx + c$ factors into $(x - \phi)(x - \hat{\phi})$, where:

$$\phi = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (4)$$

$$\hat{\phi} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (5)$$

Example

- To factor: $\mathbf{L}^2 - \mathbf{L} - 1$
- Rewrite: $1 * \mathbf{L}^2 - 1 * \mathbf{L} - 1$, $a = 1$, $b = -1$, $c = -1$
- From Quadratic Formula: $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- So $\mathbf{L}^2 - \mathbf{L} - 1$ factors to $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$

Back to Fibonnaci

- Recall the Fibonnaci recurrence is $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- We've shown the annihilator for T is $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- If we look this up in the "Lookup Table", we see that the sequence T must be of the form $\langle c_1\phi^n + c_2\hat{\phi}^n\rangle$
- All we have left to do is solve for the constants c_1 and c_2
- Can use the base cases to solve for these

Finding the Constants

- We know $T = \langle c_1\phi^n + c_2\hat{\phi}^n\rangle$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- We know

$$T(0) = c_1 + c_2 = 0 \quad (6)$$

$$T(1) = c_1\phi + c_2\hat{\phi} = 1 \quad (7)$$

- We've got two equations and two unknowns
- Can solve to get $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$,

The Punchline

- Recall Fibonnaci recurrence: $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- The final explicit formula for $T(n)$ is thus:

$$T(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(Amazingly, $T(n)$ is *always* an integer, in spite of all of the square roots in its formula.)

A Problem

- Our lookup table has a big gap: What does $(\mathbf{L}-a)(\mathbf{L}-a)$ annihilate?
- It turns out it annihilates sequences such as $\langle na^n \rangle$

Example

$$\begin{aligned}
(\mathbf{L}-a)\langle na^n\rangle &= \langle (n+1)a^{n+1} - (a)na^n\rangle \\
&= \langle (n+1)a^{n+1} - na^{n+1}\rangle \\
&= \langle (n+1-n)a^{n+1}\rangle \\
&= \langle a^{n+1}\rangle \\
(\mathbf{L}-a)^2\langle na^n\rangle &= (\mathbf{L}-a)\langle a^{n+1}\rangle \\
&= \langle 0\rangle
\end{aligned}$$

Generalization

- It turns out that $(\mathbf{L}-a)^d$ annihilates sequences of the form $\langle p(n)a^n\rangle$ where $p(n)$ is any polynomial of degree $d-1$
- Example: $(\mathbf{L}-1)^3$ annihilates the sequence $\langle n^2 * 1^n\rangle = \langle 1, 4, 9, 16, 25\rangle$ since $p(n) = n^2$ is a polynomial of degree $d-1=2$

Lookup Table

- $(\mathbf{L}-a)$ annihilates only all sequences of the form $\langle c_0 a^n \rangle$
- $(\mathbf{L}-a)(\mathbf{L}-b)$ annihilates only all sequences of the form $\langle c_0 a^n + c_1 b^n \rangle$
- $(\mathbf{L}-a_0)(\mathbf{L}-a_1)\ldots(\mathbf{L}-a_k)$ annihilates only sequences of the form $\langle c_0 a_0^n + c_1 a_1^n + \ldots c_k a_k^n \rangle$, here $a_i \neq a_j$, when $i \neq j$
- $(\mathbf{L}-a)^2$ annihilates only sequences of the form $\langle (c_0 n + c_1) a^n \rangle$
- $(\mathbf{L}-a)^k$ annihilates only sequences of the form $\langle p(n) a^n \rangle$, $degree(p(n)) = k-1$

Lookup Table

$$(\mathbf{L}-a_0)^{b_0}(\mathbf{L}-a_1)^{b_1}\ldots(\mathbf{L}-a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_1(n) a_0^n + p_2(n) a_1^n + \ldots p_k(n) a_k^n \rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)

Examples

- Q: What does $(\mathbf{L}-3)(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + c_2 3^n$
- Q: What does $(\mathbf{L}-3)^2(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + (c_2 n + c_3) 3^n$
- Q: What does $(\mathbf{L}-1)^4$ annihilate?
- A: $(c_0 n^3 + c_1 n^2 + c_2 n + c_3) 1^n$
- Q: What does $(\mathbf{L}-1)^3(\mathbf{L}-2)^2$ annihilate?
- A: $(c_0 n^2 + c_1 n + c_2) 1^n + (c_3 n + c_4) 2^n$

Annihilator Method

- Write down the annihilator for the recurrence
- Factor the annihilator
- Look up the factored annihilator in the "Lookup Table" to get general solution
- Solve for constants of the general solution by using initial conditions

Todo

- HW 1