University of New Mexico
Department of Computer Science

Final Examination

CS 362 Data Structures and Algorithms
Spring, 2024

Name:	
Email:	

Directions:

- This exam lasts 120 minutes. It is closed book and notes, and no electronic devices are permitted. However, you are allowed to use 2 pages of handwritten "cheat sheets"
- *Show your work!* You will not get full credit, if we cannot figure out how you arrived at your answer.
- Write your solution in the space provided for the corresponding problem.

Question	Points	Score	Grader
1	20		
2	20		
3	20		
4	20		
5	20		
Total	100		

1. **Short Answer (4 points each)**

   Answer the following using *simplest possible* Θ *notation.*

   (a) If n balls are dropped uniformly at random into n bins, what is the expected number of pairs of balls falling in the same bin?

   (b) Solution to the recurrence: $T(n) = 8T(n/2) + n^2$

   (c) Solution to the recurrence: $f(n) = 6f(n-1) - 9f(n-2) + 4^n$ (answer in big-O)

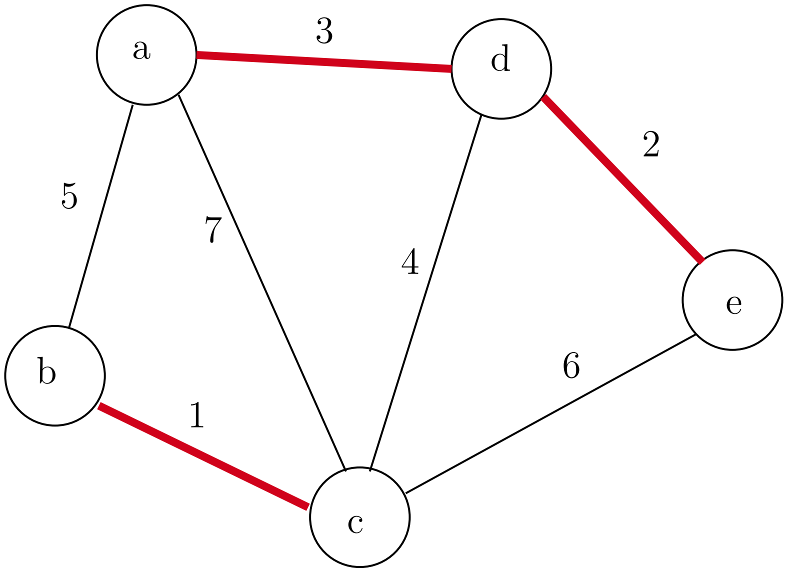
   (d) In a Union-Find data structure with n calls to *Make-Set*, and n^2 calls to all operations, what is the maximum number of calls to *Union*?

   (e) A stack has Push, Pop, and a new operation: PopUntil(x), which repeatedly pops the top item off the stack until either the stack is empty or the top item on the stack is x. What is the amortized cost per operation, over n total calls.

2. **Induction** (20 points) Consider a graph $G = (V, E)$. Call an edge *satisfied* if both endpoints of the edge are different colors. Prove that there is a way to color the nodes of G with 2 colors, such that at least half the edges are satisfied.

   Prove this by induction on the number of nodes in G. Hint: In the IS think about how to make G smaller so that you can use the IH.

3. **Safe Edges**



(a) (10 points) Let A be the set of bold, red edges in the above graph. Give a cut that respects A, and the corresponding light edge crossing that cut.

Let A be a subset of edges in a minimum spanning tree(MST). Recall that an edge, e is said to be *safe* for A if $A \cup \{e\}$ is also a subset of edges in a MST. The safe edge theorem says: "If e is a light edge that crosses some cut that respects A, then e is safe for A."

(b) (10 points) Is the following statement true: "Consider any cut that respects A. If e is a safe edge for A that crosses that cut, then e is a light edge crossing the cut"? If so, give a proof. If not, give a counterexample.

4. **Cable Cutting**

   You have n feet of cable to be cut it into pieces for resale. On a given day, pieces of length 1, 3, and 7 can resell for values of v_1, v_2 and v_3. Your want to cut the cable into pieces with maximum total resell value.

   For example, if $n = 14$, and $v_1 = 1$, $v_2 = 4$ and $v_3 = 8$, then the optimal cutting is: 4 pieces of length 3, and 2 pieces of length 1, for a total resell value of $4 * 4 + 2 * 1 = 18$.

   (a) (10 points) You decide to solve this problem with dynamic programming. For any number $i \in [0, n]$, let $m(i)$ be the maximum resell value you can get from optimally cutting a cable of length i. Write a recurrence relation for $m(i)$. Don't forget the base case(s)

Now consider a variant of the problem where the maximum number of cuts you can make is some integer k. As before, pieces of length 1, 3, and 7 resell for values of v_1, v_2 and v_3. Any pieces of other lengths have zero value. For example, if $n = 14$, $k = 1$ and $v_1 = 1$, $v_2 = 4$ and $v_3 = 8$, then the optimal cutting is: 2 pieces of length 7 for a total resell value of $2 * 8 = 16$.

(b) (10 points) Write a recurrence relation for a dynamic program for this variant. In particular, for any numbers $i \in [0, n]$ and $j \in [0, k]$, let $m(i, k)$ be the maximum resell value you can get from cutting a cable of length i with at most k cuts. Write a recurrence relation for $m(i, k)$. Don't forget the base case(s).

5. **MAX-INSIDE-EDGES** For a given set of nodes S in a graph G, call an edge of G *inside* S if both endpoints of the edge are nodes in S.

   In the MAX-INSIDE-EDGES problem, you are given a graph $G = (V, E)$, and a number x. You must output the *maximum* number of inside edges in any set S such that $S \subseteq V$ and $|S| \leq x$.

   (a) (10 points) Prove that MAX-INSIDE-EDGES is NP-Hard by a reduction from one of the following: 3-SAT, VERTEX-COVER, CLIQUE, SUBGRAPH-ISOMORPHISM, INDEPENDENT-SET, 3-COLORABLE, HAMILTONIAN-CYCLE, or TSP.

(b) (10 points) Consider the randomized algorithm that picks a subset S of size x, uniformly at random from all subsets of size x. Compute the expected number of inside edges for this algorithm. Let $n = |V|$ and $m = |E|$. Hint: Use indicator random variables and linearity of expectation.