CS 362, HW 10

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1. Professor Curly conjectures that if we do union by rank, *without path compression*, the amortized cost of all operations is $o(\log n)$. Prove him wrong by showing that if we do union by rank without path compression, there can be m MAKESET, UNION and FINDSET operations, n of which are MAKESET operations, where the total cost of all operations is $\Theta(m \log n)$.

2. Consider a connected graph $G = (V, E)$. Call a subset of edges, F, a *cycle cover* if every cycle in G contains at least one edge in F. In other words, removing the edges of F from G results in an acyclic graph. You want to find a cycle cover, F, of G with *minimum* weight, i.e. the sum of the weight of all edges in F is minimized over all cycle covers. Give an efficient algorithm to solve this, and give the runtime of your algorithm as a function of $n = |V|$ and $m = |E|$. Hint: Think about the maximum-weight spanning tree problem.

3. Professor Matsumoto conjectures the following converse of the safe-edge theorem:
   Let $G = (V, E)$ be a connected, undirected, weighted graph, with weight function w. Let A be a subset of E that is included in some minimum spanning tree of G. Let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a safe edge for A that crosses $(S, V - S)$. Then (u, v) is a light edge for the cut.
   Is this conjecture true? If so, prove it. If not, give a counterexample.

4. Prove that if an edge (u, v) is in some minimum spanning tree for a graph G, then (u, v) is a light edge crossing some cut in G.