All Pairs Shortest Paths

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Outline

- All Pairs Shortest Paths
- Floyd Warshall Algorithm

All-Pairs Shortest Paths

- For the single-source shortest paths problem, we wanted to find the shortest path from a source vertex s to all the other vertices in the graph
- We will now generalize this problem further to that of finding the shortest path from *every* possible source to *every* possible destination
- In particular, for every pair of vertices u and v, we need to compute the following information:
  - $dist(u, v)$ is the length of the shortest path (if any) from u to v
  - $pred(u, v)$ is the second-to-last vertex (if any) on the shortest path (if any) from u to v

Example

- For any vertex v, we have $dist(v, v) = 0$ and $pred(v, v) = NULL$
- If the shortest path from u to v is only one edge long, then $dist(u, v) = w(u \rightarrow v)$ and $pred(u, v) = u$
- If there's no shortest path from u to v, then $dist(u, v) = \infty$ and $pred(u, v) = NULL$

APSP

- The output of our shortest path algorithm will be a pair of $n \times n$ arrays encoding all n^2 distances and predecessors.
- Many maps contain such a distance matric - to find the distance from (say) Albuquerque to (say) Ruidoso, you look in the row labeled “Albuquerque” and the column labeled “Ruidoso”
- In this class, we’ll focus only on computing the distance array
- The predecessor array, from which you would compute the actual shortest paths, can be computed with only minor additions to the algorithms presented here

Lots of Single Sources

- Most obvious solution to APSP is to just run SSSP algorithm n times, once for every possible source vertex
- Specifically, to fill in the subarray $dist(s, *)$, we invoke either Dijkstra's or Bellman-Ford starting at the source vertex s
- We'll call this algorithm ObviousAPSP

ObviousAPSP

```
ObviousAPSP(V,E,w){
  for every vertex s{
    dist(s,*) = SSSP(V,E,w,s);
  }
}
```

Analysis

- The running time of this algorithm depends on which SSSP algorithm we use
- If we use Bellman-Ford, the overall running time is $O(n^2m) = O(n^4)$
- If all the edge weights are positive, we can use Dijkstra's instead, which decreases the run time to $\Theta(nm + n^2 \log n) = O(n^3)$

Problem

- We'd like to have an algorithm which takes $O(n^3)$ but which can also handle negative edge weights
- We'll see that a dynamic programming algorithm, the Floyd Warshall algorithm, will achieve this
- Note: the book discusses another algorithm, Johnson's algorithm, which is asymptotically better than Floyd Warshall on sparse graphs. However we will not be discussing this algorithm in class.

Dynamic Programming

- Recall: Dynamic Programming = Recursion + Memorization
- Thus we first need to come up with a recursive formulation of the problem
- We might recursively define $dist(u, v)$ as follows:

$$dist(u,v) = \begin{cases} 0 & \text{if } u = v \\ \min_x \Big(dist(u,x) + w(x \to v) \Big) & \text{otherwise} \end{cases}$$

The problem

- In other words, to find the shortest path from u to v, try all possible predecessors x, compute the shortest path from u to x and then add the last edge $u \to v$
- **Unfortunately, this recurrence doesn't work**
- To compute $dist(u, v)$, we first must compute $dist(u, x)$ for every other vertex x, but to compute any $dist(u, x)$, we first need to compute $dist(u, v)$
- We're stuck in an infinite loop!

The solution

- To avoid this circular dependency, we need some additional parameter that decreases at each recursion and eventually reaches zero at the base case
- One possibility is to include the number of edges in the shortest path as this third magic parameter
- So define $dist(u, v, k)$ to be the length of the shortest path from u to v that uses *at most* k edges
- Since we know that the shortest path between any two vertices uses at most $n - 1$ edges, what we want to compute is $dist(u, v, n - 1)$

Edge-Hop Recurrence (SLOW)

$$dist(u,v,k) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } k = 0 \text{ and } u \neq v \\ \min_x \Big(dist(u,x,k-1) + w(x \to v) \Big) & \text{otherwise} \end{cases}$$

Edge-Hop DP (SLOW)

- It's not hard to turn this recurrence into a dynamic programming algorithm
- Even before we write down the algorithm, though, we can tell that its running time will be $\Theta(n^4)$
- This is just because the recurrence has four variables — u, v, k and x — each of which can take on n different values
- Except for the base cases, the algorithm will just be four nested "for" loops

The Problem

- This algorithm still takes $O(n^4)$ which is no better than the ObviousAPSP algorithm
- If we use a certain divide and conquer technique, there is a way to get this down to $O(n^3 \log n)$ (think about how you might do this)
- However, to get down to $O(n^3)$ run time, we need to use a different third parameter in the recurrence

Floyd-Warshall

- Number the vertices arbitrarily from 1 to n
- Define $dist(u, v, r)$ to be the shortest path from u to v where all *intermediate* vertices (if any) are numbered r or less
- If $r = 0$, we can't use any intermediate vertices so shortest path from u to v is just the weight of the edge (if any) between u and v
- If $r > 0$, then either the shortest legal path from u to v goes through vertex r or it doesn't
- We need to compute the shortest path distance from u to v with no restrictions, which is just $dist(u, v, n)$

The Floyd-Warshall Recurrence

We get the following recurrence:

$$dist(u,v,r) = \begin{cases} w(u \to v) & \text{if } r = 0 \\ \min\{dist(u,v,r-1), & \\ \qquad dist(u,r,r-1) + dist(r,v,r-1)\} & \text{otherwise} \end{cases}$$

The Algorithm

```
FloydWarshall(V,E,w){
  for u=1 to n{
    for v=1 to n{
      dist(u,v,0) = w(u,v);
  }}
  for r=1 to n{
    for u=1 to n{
      for v=1 to n{
        if (dist(u,v,r-1) < dist(u,r,r-1) + dist(r,v,r-1))
          dist(u,v,r) = dist(u,v,r-1);
        else
          dist(u,v,r) = dist(u,r,r-1) + dist(r,v,r-1);
}}}}
```

Analysis

- There are three variables here, each of which takes on n possible values
- Thus the run time is $\Theta(n^3)$
- Space required is also $\Theta(n^3)$

Take Away

- Floyd-Warshall solves the APSP problem in $\Theta(n^3)$ time even with negative edge weights
- Floyd-Warshall uses dynamic programming to compute APSP
- We've seen that sometimes for a dynamic program, we need to introduce an *extra variable* to break dependencies in the recurrence.
- We've also seen that the choice of this extra variable can have a big impact on the run time of the dynamic program