University of New Mexico
Department of Computer Science

Midterm Examination

CS 362 Data Structures and Algorithms
Spring, 2025

Name:	
Email:	

Directions:

- This exam lasts 75 minutes. It is closed book and notes, and no electronic devices are permitted. However, you are allowed to use 2 pages of "cheat sheets"
- *Show your work!* You will not get full credit, if we cannot figure out how you arrived at your answer.
- Write your solution in the space provided for the corresponding problem.

Question	Points	Score	Grader
1	20		
2	20		
3	20		
4	20		
5	20		
Total	100		

1. **Short Answer (4 points each)**

   Answer the following using *simplest possible* Θ *notation.*

   (a) Expected number of items at the 4-th level (from the bottom) of a skip list containing n items?

   (b) Solution to the recurrence: $T(n) = 4T(n/4) + n$

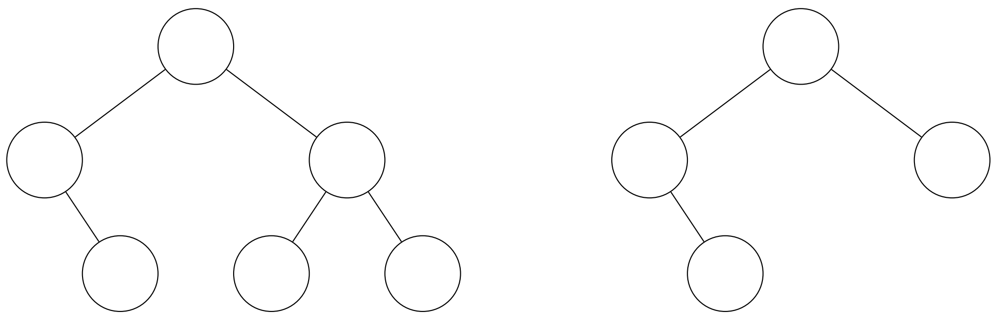
   (c) Solution to the recurrence: $T(n) = 8T(n/2) + \sqrt{n}$

   (d) Solution to the recurrence: $f(n) = 4f(n-1) - 3f(n-2) + 3^n$ (answer in big-O)

   (e) Solution to the recurrence $F(n) = 2F(\sqrt{n}) + \log n$ (answer in big-O).

2. **Induction (20 points)**

In a *node-balanced binary tree*, the size of the left and right sub-trees of any node differ by at most 1, where the *size* of a tree is the number of nodes it contains. Below are two node-balanced binary trees.



(20 points) Recall that the height, h, of a tree is the maximum number of edges in any path from the root to a leaf. Prove that the size of a node-balanced binary tree of height h is always at least 2^h. Prove this by induction on h. Don't forget the BC, IH and IS.

3. **Probability** There is a grid with n by n nodes, n^2 **nodes total**. Each node is independently colored green with probability p and red otherwise.

   (a) (4 points) What is the expected number of green nodes?

   (b) (4 points) Use Markov's inequality to get an upper bound on the probability that at least one node is green.

   (c) (4 points) Now use a union bound to bound the probability that no node is green.

   (d) (8 points) A node is an *interior* node if it has 4 neighbors. So, our grid has $(n-2)^2$ interior nodes. Now, let $p = 1/2$. What is the expected number of interior nodes where the node and all 4 of its neighbors have the same color?

4. **The Creaky Staircase**

On a dark and quiet night, you find yourself climbing a creaky staircase of n stairs. Every stair, $i \in [1, n]$ has a cost value c_i measuring its "creakiness". In each step, you can go up either 2 or 3 stairs, and your goal is to reach the top with minimum total stair cost. The bottom and top of the staircase are not creaky and so have no cost. On the last two stairs, either step size will take you to the top.

For example, if $n = 4$ and the costs are $[2, 1, 5, 8]$, you can first go up 2 stairs at a cost of 1, then go up 3 stairs at a cost of 0, for total cost of 1.

(a) (15 points) Write a recurrence relation to solve this problem. Don't forget to first define the function in words whose solutions enable solving the big problem.

(b) (5 points) Describe a dynamic program for this problem using your recurrence. What are the dimensions of your table? How do you fill it? What do you return as the final answer? What is the runtime of your algorithm?

5. **Game of NIM.** NIM is a two-player game with 3 piles of stones. Players take turns removing any positive number of stones from any pile they choose. Whoever takes the last stone loses. Below is a sample game starting with piles of 13, 9, and 1, where Player 1 first removes the single stone from pile 3 and eventually wins.

Player Turn	1	2	1	2	1	2
Stones left	(13, 9, 1)	(13, 9, 0)	(13, 2, 0)	(2, 2, 0)	(1, 2, 0)	(1, 0, 0)

In this problem, you will write a dynamic program to determine if the current player can force a win for a given input, where the input is the size of each of the 3 piles.

(a) (2 points) Describe in words a function whose solutions for smaller problems will help you solve the big problem.

(b) (13 points) Write a recurrence relation for this function.

(c) (5 points) Describe a dynamic program to solve the problem for any initial input consisting of x_1, x_2, x_3 stones in the piles. What are the dimensions of your table? How do you fill it in? What is the final value returned? What is the runtime of your algorithm if $x_1 = x_2 = x_3 = n$?