CS 561, Lecture 19

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Today's Outline

- Data Structures for Disjoint Sets

Disjoint Sets

- A disjoint set data structure maintains a collection $\{S_1, S_2, \ldots S_k\}$ of disjoint dynamic sets
- Each set is identified by a representative which is a member of that set
- Let's call the members of the sets *objects*.

Operations

We want to support the following operations:

- Make-Set(x): creates a new set whose only member (and representative) is x
- Union(x,y): unites the sets that contain x and y (call them S_x and S_y) into a new set that is $S_x \cup S_y$. The new set is added to the data structure while S_x and S_y are deleted. The representative of the new set is any member of the set.
- Find-Set(x): Returns a pointer to the representative of the (unique) set containing x

Analysis

- We will analyze this data structure in terms of two parameters:
  1. n, the number of Make-Set operations
  2. m, the total number of Make-Set, Union, and Find-Set operations
- Since the sets are always disjoint, each Union operation reduces the number of sets by 1
- So after $n-1$ Union operations, only one set remains
- Thus the number of Union operations is at most $n-1$

Analysis

- Note also that since the Make-Set operations are included in the total number of operations, we know that $m \geq n$
- We will in general assume that the Make-Set operations are the first n performed

Application

- Myspace is a web site which keeps track of a social network
- When you are invited to join Myspace, you become part of the social network of the person who invited you to join
- In other words, you can read profiles of people who are friends of your initial friend, or friends of friends of your initial friend, etc., etc.
- If you forge links to new people in Myspace, then your social network grows accordingly

Application

- Consider a simplified version of Myspace
- Every object is a person and every set represents a social network
- Whenever a person in the set S_1 forges a link to a person in the set S_2, then we want to create a new larger social network $S_1 \cup S_2$ (and delete S_1 and S_2)
- For obvious reasons, we want these operation of Union, Make-Set and Find-Set to be as fast as possible

Example

- Make-Set("Bob"), Make-Set("Sue"), Make-Set("Jane"), Make-Set("Joe")
- Union("Bob", "Joe")
  there are now three sets $\{Bob, Joe\}, \{Jane\}, \{Sue\}$
- Union("Jane", "Sue")
  there are now two sets $\{Bob, Joe\}, \{Jane, Sue\}$
- Union("Bob","Jane")
  there is now one set $\{Bob, Joe, Jane, Sue\}$

Applications

- We will also see that this data structure is used in Kruskal's minimum spanning tree algorithm
- Another application is maintaining the connected components of a graph as new vertices and edges are added

Tree Implementation

- One of the easiest ways to store sets is using trees.
- Each object points to another object, called its *parent*, except for the leader of each set, which points to itself and thus is the root of the tree.

Tree Implementation

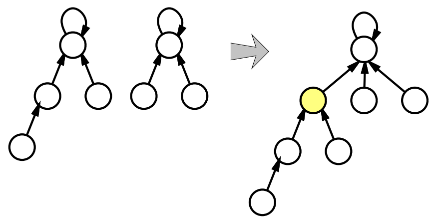
- Make-Set is trivial (we just create one root node)
- Find-Set traverses the parent pointers up to the leader (the root node).
- Union just redirects the parent pointer of one leader to the other.

(Notice that unlike most tree data structures, objects do *not* have pointers down to their children.)

Algorithms

```
Make-Set(x){
  parent(x) = x;
}
Find-Set(x){
  while(x!=parent(x))
    x = parent(x);
  return x;
}
Union(x,y){
  xParent = Find-Set(x);
  yParent = Find-Set(y);
  parent(yParent) = xParent;
}
```


Example



Merging two sets stored as trees. Arrows point to parents. The shaded node has a new parent.

Analysis

- Make-Set takes $\Theta(1)$ time
- Union takes $\Theta(1)$ time in addition to the calls to Find-Set
- The running time of Find-Set is proportional to the depth of x in the tree. In the worst case, this could be $\Theta(n)$ time

Problem

- Problem: The running time of Find-Set is very slow
- Q: Is there some way to speed this up?
- A: Yes we can ensure that the depths of our trees remain small
- We can do this by using the following strategy when merging two trees: we make the root of the tree with fewer nodes a child of the tree with more nodes
- This means that we need to always store the number of nodes in each tree, but this is easy

The Code

```
Make-Set(x){
  parent(x) = x;
  size(x) = 1;
}
Union(x,y){
  xRep = Find-Set(x);
  yRep = Find-Set(y);
  if (size(xRep)) > size(yRep)){
    parent(yRep) = xRep;
    size(xRep) = size(xRep) + size(yRep);
  }else{
    parent(xRep) = yRep;
    size(yRep) = size(yRep) + size(xRep);
    }
  }
```


Analysis

- It turns out that for these algorithms, all the functions run in $O(\log n)$ time
- We will be showing this is the case in the In-Class exercise
- We will show this by showing that the heights of all the trees are always logarithmic in the number of nodes in the tree

In-Class Exercise

- We will show that the depth of our trees are no more than $O(\log x)$ where x is the number of nodes in the tree
- We will show this using proof by induction on, x, the number of nodes in the tree
- We will consider a tree with x nodes and, using the inductive hypothesis (and facts about our algs), show that it has a height of of $O(\log x)$

The Facts

- Let T be a tree with x nodes that was created by a call to the Union Algorithm
- Note that T must have been created by merging two trees $T1$ and $T2$
- Let $T2$ be the tree with the smaller number of nodes
- Then the root of T is the root of $T1$ and a child of this root is the root of the tree $T2$
- Key fact: the number of nodes in $T2$ is no more than $x/2$

In-Class Exercise

To prove: Any tree T with x nodes, created by our algorithms, has depth no more than $\log x$

- Q1: Show the base case ($x = 1$)
- Q2: What is the inductive hypothesis?
- Q3: Complete the proof by giving the inductive step. (hint: note that depth(T) = Max(depth(T1),depth(T2)+1)

Problem

- Q: $O(\log n)$ per operation is not bad but can we do better?
- A: Yes we can actually do much better but it's going to take some cleverness (and amortized analysis)

Shallow Threaded Trees

- One good idea is to just have every object keep a pointer to the leader of it's set
- In other words, each set is represented by a tree of depth 1
- Then Make-Set and Find-Set are completely trivial, and they both take $O(1)$ time
- Q: What about the Union operation?

Union

- To do a union, we need to set all the leader pointers of one set to point to the leader of the other set
- To do this, we need a way to visit all the nodes in one of the sets
- We can do this easily by "threading" a linked list through each set starting with the sets leaders
- The threads of two sets can be merged by the Union algorithm in constant time

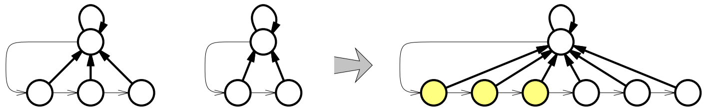
The Code

```
Make-Set(x){
  leader(x) = x;
  next(x) = NULL;
}
Find-Set(x){
  return leader(x);
}
```


The Code

```
Union(x,y){
  xRep = Find-Set(x);
  yRep = Find-Set(y);
  leader(y) = xRep;
  while(next(y)!=NULL){
    y = next(y);
    leader(y) = xRep;
  }
  next(y) = next(xRep);
  next(xRep) = yRep;
}
```


Example



Merging two sets stored as threaded trees.
Bold arrows point to leaders; lighter arrows form the threads.
Shaded nodes have a new leader.

Analysis

- Worst case time of Union is a constant times the size of the *larger* set
- So if we merge a one-element set with a n element set, the run time can be $\Theta(n)$
- In the worst case, it's easy to see that n operations can take $\Theta(n^2)$ time for this alg

Problem

- The main problem here is that in the worst case, we always get unlucky and choose to update the leader pointers of the larger set
- Instead let's purposefully choose to update the leader pointers of the smaller set
- This will require us to keep track of the sizes of all the sets, but this is not difficult

The Code

```
Make-Weighted-Set(x){
  leader(x) = x;
  next(x) = NULL;
  size(x) = 1;
}
```


The Code

```
Weighted-Union(x,y){
  xRep = Find-Set(x);
  yRep = Find-Set(y)
  if(size(xRep)>size(yRep){
    Union(xRep,yRep);
    size(xRep) = size(xRep) + size(yRep);
  }else{
    Union(yRep,xRep);
    size(yRep) = size(xRep) + size(yRep);
  }
}
```


Analysis

- The Weighted-Union algorithm still takes $\Theta(n)$ time to merge two n element sets
- However in an amortized sense, it is more efficient:
- A sequence of m Make-Weighted-Set operations and n Weighted-Union operations takes $O(m+n\log n)$ time in the worst case.