CS 561, Lecture 8

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Today's Outline

- Hash Tables
- Trees

Direct Addressing Problem

- If universe U is large, storing the array T may be impractical
- Also much space can be wasted in T if number of objects stored is small
- Q: Can we do better?
- A: Yes we can trade time for space

Hash Tables

- "Key" Idea: An element with key k is stored in slot $h(k)$, where h is a *hash function* mapping U into the set $\{0, \ldots, m-1\}$
- Main problem: Two keys can now hash to the same slot
- Q: How do we resolve this problem?
- A1: Try to prevent it by hashing keys to "random" slots and making the table large enough
- A2: Chaining
- A3: Open Addressing

Chained Hash

In chaining, all elements that hash to the same slot are put in a linked list.

```
CH-Insert(T,x){Insert x at the head of list T[h(key(x))];}
CH-Search(T,k){search for elem with key k in list T[h(k)];}
CH-Delete(T,x){delete x from the list T[h(key(x))];}
```


Analysis

- CH-Insert and CH-Delete take $O(1)$ time if the list is doubly linked and there are no duplicate keys
- Q: How long does CH-Search take?
- A: It depends. In particular, depends on the *load factor*, $\alpha = n/m$ (i.e. average number of elems in a list)

CH-Search Analysis

- Worst case analysis: everyone hashes to one slot so $\Theta(n)$
- For average case, make the *simple uniform hashing* assumption: any given elem is equally likely to hash into any of the m slots, indep. of the other elems
- Let n_i be a random variable giving the length of the list at the i-th slot
- Then time to do a search for key k is $1 + n_{h(k)}$

CH-Search Analysis

- Q: What is $E(n_{h(k)})$?
- A: We know that $h(k)$ is uniformly distributed among $\{0, .., m-1\}$
- Thus, $E(n_{h(k)}) = \sum_{i=0}^{m-1}(1/m)n_i = n/m = \alpha$

Hash Functions

- Want each key to be equally likely to hash to any of the m slots, independently of the other keys
- Key idea is to use the hash function to "break up" any patterns that might exist in the data
- We will always assume a key is a natural number (can e.g. easily convert strings to naturaly numbers)

Division Method

- $h(k) = k \mod m$
- Want m to be a *prime number*, which is not too close to a power of 2
- Why?

Multiplication Method

- $h(k) = \lfloor m * (kA \mod 1) \rfloor$
- $kA \mod 1$ means the fractional part of kA
- Advantage: value of m is not critical, need not be a prime
- $A = (\sqrt{5} - 1)/2$ works well in practice

Open Addressing

- All elements are stored in the hash table, there are no separate linked lists
- When we do a search, we probe the hash table until we find an empty slot
- Sequence of probes depends on the key
- Thus hash function maps from a key to a "probe sequence" (i.e. a permutation of the numbers $0, .., m - 1$)

Open Addressing

- In general, for open addressing, the hash function depends on both the key to be inserted and the *probe number*
- Thus for a key k, we get the probe sequence $h(k, 0), h(k, 1), \ldots, h(k, m-1)$

Open Addressing

- If we use open addressing, the hash table can never fill up i.e. the load factor α can never exceed 1
- An advantage of open addressing is that it avoids pointers and the overhead of storing lists in each slot of the table
- This freed up memory can be used to create more slots in the table which can reduce the load-factor and potentially speed up retrieval time
- A disadvantage is that deletion is difficult. If deletions occur in the hash table, chaining is usually used

OA-Insert

```
OA-Insert(T,k){
  i = 0;
  repeat {
    j = h(k,i);
    if (T[j] = nil){
      T[j] = k;
      return j;
    }
    else i++;
  } until (i==m);

}
```


OA-Search

```
OA-Insert(T,k){
  i = 0;
  repeat {
    j = h(k,i);
    if (T[j] = k){
      return j;
    }
    else i++;
  } until (T[j]==nil or i==m);
}
```

OA-Delete

- Deletion from an open-address hash table is difficult
- When we delete a key from slot i, we can't just mark that slot as empty by storing nil there
- The problem is that this would make it impossible to find some key k during whose insertion we probed slot i and found it occupied

OA-Delete

- One solution is to mark the slot by storing in it the value "DELETED"
- Then we modify OA-Insert to treat such a slot as if it were empty so that something can be stored in it
- OA-Search passes over these special slots while searching
- Note that if we use this trick, search times are no longer dependent on the load-factor α (for this reason, chaining is more commonly used when keys must be deleted)

Implementation

- To analyze open-address hashing, we make the assumption of *uniform hashing*: we assume that each key is equally likely to have any of the $m!$ permutations of $\{0, 1, \ldots, m-1\}$ as its probe sequence
- True uniform hashing is difficult to implement, so in practice, we generally use one of three approximations on the next slide

Implementations

All positions are taken modulo m, and i ranges from 1 to $m-1$

- *Linear Probing*: Initial probe is to position $h(k)$, successive probes are to positions $h(k)+i$,
- *Quadratic Probing*: Initial probes is to position $h(k)$, successive probes are to position $h(k)+c_1 i + c_2 i^2$
- *Double Hashing*: Initial probe is to position $h(k)$, successive probes are to positions $h(k)+ih_2(k)$

Analysis

- Recall that the load factor, α, is the number of elements stored in the hash table, n, divided by the total number of slots m
- In open-address hashing, we have at most one element per slot so $\alpha < 1$
- We assume uniform hashing i.e. each probe maps to essentially a random slot in the table.
- We can show that the expected time for insertions is at most $1/(1-\alpha)$, the expected time for an unsuccessful search is $1/(1-\alpha)$ and the expected time for a successful search is $(1/\alpha)\ln[1/(1-\alpha)]$

Hash Tables Wrapup

Hash Tables implement the Dictionary ADT, namely:

- Insert(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Lookup(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Delete(x) - $O(1)$ expected time, $\Theta(n)$ worst case

Binary Search Trees

- Binary Search Trees are another data structure for implementing the dictionary ADT

Red-Black Trees

Red-Black trees (a kind of binary tree) also implement the Dictionary ADT, namely:

- Insert(x) - $O(\log n)$ time
- Lookup(x) - $O(\log n)$ time
- Delete(x) - $O(\log n)$ time

Why BST?

- Q: When would you use a Search Tree?
- A1: When need a hard guarantee on the worst case run times (e.g. "mission critical" code)
- A2: When want something more dynamic than a hash table (e.g. don't want to have to enlarge a hash table when the load factor gets too large)
- A3: Search trees can implement some other important operations...

Search Tree Operations

- Insert
- Lookup
- Delete
- *Minimum/Maximum*
- *Predecessor/Successor*

What is a BST?

- It's a binary tree
- Each node holds a key and record field, and a pointer to left and right children
- *Binary Search Tree Property* is maintained

Binary Search Tree Property

- Let x be a node in a binary search tree. If y is a node in the left subtree of x, then key(y)≤key(x). If y is a node in the right subtree of x then key(x)≤key(y)

Example BST

Inorder Walk

- BSTs are arranged in such a way that we can print out the elements in sorted order in $\Theta(n)$ time
- Inorder Tree-Walk does this

Inorder Tree-Walk

```
Inorder-TW(x){
  if (x is not nil){
    Inorder-TW(left(x));
    print key(x);
    Inorder-TW(right(x));
}
```


Example Tree-Walk

Analysis

- Correctness?
- Run time?

Search in BT

```
Tree-Search(x,k){
  if (x=nil) or (k = key(x)){
    return x;
  }
  if (k<key(x)){
    return Tree-Search(left(x),k);
  }else{
    return Tree-Search(right(x),k);
  }
}
```


Analysis

- Let h be the height of the tree
- The run time is $O(h)$
- Correctness???

In-Class Exercise

- Q1: What is the loop invariant for Tree-Search?
- Q2: What is Initialization?
- Q3: Maintenance?
- Q4: Termination?