CS 561, HW 10

Prof. Jared Saia, University of New Mexico

1. Prove the following fact, from Slide 114, amortized lecture, about the Union-Find data structure using PC-Union:

   - For any set X, size(X) $\geq 2^{rank(leader(X))}$

   Prove this by induction on the size of X. Don't forget to include the BC, IH and IS.

2. Professor Moe conjectures that for any connected graph G, the set of edges $\{$(u,v) : there exists a cut (S,V-S) such that (u,v) is a light edge crossing (S, V-S)$\}$ always forms a minimum spanning tree. Given a simple example of a connected graph that proves him wrong.

3. Consider a connected graph $G = (V, E)$. Call a subset of edges, F, a *cycle cover* if every cycle in G contains at least one edge in F. In other words, removing the edges of F from G results in an acyclic graph. You want to find a cycle cover, F, of G with *minimum* weight, i.e. the sum of the weight of all edges in F is minimized over all cycle covers. Give an efficient algorithm to solve this, and give the runtime of your algorithm as a function of $n = |V|$ and $m = |E|$. Hint: Think about the maximum-weight spanning tree problem.

4. Professor Matsumoto conjectures the following converse of the safe-edge theorem:
   Let $G = (V, E)$ be a connected, undirected, weighted graph, with weight function w. Let A be a subset of E that is included in some minimum spanning tree of G. Let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a safe edge for A that crosses $(S, V - S)$. Then (u, v) is a light edge for the cut.
   Is this conjecture true? If so, prove it. If not, give a counterexample.

5. Prove that if an edge (u, v) is in some minimum spanning tree for a graph G, then (u, v) is a light edge crossing some cut in G.