Recurrences and Induction (Review)

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Today's Outline

- Recurrence Relations
- Induction

Recurrence Relations

"Oh how should I not lust after eternity and after the nuptial ring of rings, the ring of recurrence" - Friedrich Nietzsche, Thus Spoke Zarathustra

- Getting the run times of recursive algorithms can be challenging
- Consider an algorithm for binary search (next slide)
- Let $T(n)$ be the run time of this algorithm on an array of size n
- Then we can write $T(1) = 1$, $T(n) = T(n/2) + 1$

Binary Search

Assume A is sorted; low and $high$ are inclusive.

BinarySearch$(A, key, low, high)$

if $low > high$ **then return** false
$mid \leftarrow \lfloor (low + high)/2 \rfloor$
if $A[mid] = key$ **then return** true
if $key < A[mid]$
  return BinarySearch$(A, key, low, mid - 1)$
else
  return BinarySearch$(A, key, mid + 1, high)$

Initial call: BinarySearch$(A, key, 0, n - 1)$.

Recurrence Relations

- $T(n) = T(n/2) + 1$ is an example of a *recurrence* relation
- A *Recurrence Relation* is any equation for a function T, where T appears on both the left and right sides of the equation.
- We always want to "solve" these recurrence relation by getting an equation for T, where T appears on just the left side of the equation

Recurrence Relations

- Whenever we analyze the run time of a recursive algorithm, we will first get a recurrence relation
- To get the actual run time, we need to solve the recurrence relation

Substitution Method

- One way to solve recurrences is the substitution method aka "guess and check"
- What we do is make a good guess for the solution to $T(n)$, and then try to prove this is the solution by induction

Example

- Let's guess that the solution to $T(n) = T(n/2) + 1$, $T(1) = 1$ is $T(n) = O(\log n)$
- In other words, $T(n) \leq c \log n$ for all $n \geq n_0$, for some positive constants c, n_0
- We can prove that $T(n) \leq c \log n$ is true by plugging back into the recurrence

Proof

We prove this by induction:

- BC: $T(2) = 2 \leq c \log 2$ provided that $c \geq 2$
- IH: For all $j < n$, $T(j) \leq c \log(j)$
- IS:

$$\begin{aligned} T(n) &= T(n/2) + 1 \\ &\leq c \log(n/2) + 1 && \text{By IH} \\ &= c(\log n - \log 2) + 1 \\ &= c \log n - c + 1 \\ &\leq c \log n \end{aligned}$$

Last step holds for all $n > 0$ if $c \geq 1$. Thus, entire proof holds if $n \geq 2$ and $c \geq 2$.

Some Examples

- The next four problems can be attacked by induction/recurrences
- For each problem, we'll need to reduce it to smaller problems
- Question: How can we reduce each problem to a smaller subproblem?

(1) Sum Problem

- $f(n)$ is the sum of the integers $1, \ldots, n$

(2) Tree Problem

- $f(n)$ is the maximum number of leaf nodes in a binary tree of height n

Recall:

- In a binary tree, each node has at most two children
- A *leaf* node is a node with no children
- The height of a tree is the length of the longest path from the root to a leaf node.

(3) Binary Search Problem

- $f(n)$ is the maximum number of queries that need to be made for binary search on a sorted array of size n.

(4) Dominoes Problem

- $f(n)$ is the number of ways to tile a 2 by n rectangle with dominoes (a domino is a 2 by 1 rectangle)
- $f(0) = 1$ (the empty tiling) and $f(1) = 1$

Simpler Subproblems

- Sum Problem: What is the sum of all numbers between 1 and $n-1$ (i.e. $f(n-1)$)?
- Tree Problem: What is the maximum number of leaf nodes in a binary tree of height $n-1$? (i.e. $f(n-1)$)
- Binary Search Problem: What is the maximum number of queries that need to be made for binary search on a sorted array of size $n/2$? (i.e. $f(n/2)$)
- Dominoes problem: What is the number of ways to tile a 2 by $n-1$ rectangle with dominoes? What is the number of ways to tile a 2 by $n-2$ rectangle with dominoes? (i.e. $f(n-1)$, $f(n-2)$)

Recurrences

- Sum Problem: $f(n) = f(n-1) + n$, $f(1) = 1$
- Tree Problem: $f(n) = 2f(n-1)$, $f(0) = 1$
- Binary Search Problem: $f(n) = f(n/2) + 1$, $f(1) = 1$
- Dominoes problem: $f(n) = f(n-1) + f(n-2)$, $f(0) = 1$, $f(1) = 1$

Guesses

- Sum Problem: $f(n) = (n+1)n/2$
- Tree Problem: $f(n) = 2^n$
- Binary Search Problem: $f(n) = \log_2 n + 1$
- Dominoes problem: $f(n) = (\phi^{n+1} - \psi^{n+1})/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$

Inductive Proofs

"Trying is the first step to failure" - *Homer Simpson*

- Now that we've made these guesses, we can try using induction to prove they're correct (the substitution method)
- We'll give inductive proofs that these guesses are correct for the first three problems

Sum Problem

- Want to show that $f(n) = (n+1)n/2$.
- Prove by induction on n
- BC: $f(1) = 2 * 1/2 = 1$
- IH: for all $j < n$, $f(j) = (j+1)j/2$
- IS:

$$
\begin{aligned}
f(n) &= f(n-1) + n \\
&= n(n-1)/2 + n && \text{By the IH} \\
&= (n+1)n/2
\end{aligned}
$$

Tree Problem

- Want to show that $f(n) = 2^n$.
- Prove by induction on n
- BC: $f(0) = 2^0 = 1$
- IH: for all $j < n$, $f(j) = 2^j$
- IS:

$$\begin{aligned} f(n) &= 2 \cdot f(n-1) & \\ &= 2 \cdot 2^{n-1} & \text{by the IH} \\ &= 2^n & \end{aligned}$$

Binary Search Problem

- Want to show that $f(n) = \log_2 n + 1$ (assume n is a power of 2)
- Write $n = 2^k$ and prove the claim by induction on k
- BC ($k = 0$): $f(1) = \log_2 1 + 1 = 1$
- IH: $f(2^{k-1}) = k$
- IS (for $k \geq 1$):

$$
\begin{aligned}
f(2^k) &= f(2^{k-1}) + 1 \\
&= k + 1 && \text{by the IH} \\
&= \log_2(2^k) + 1
\end{aligned}
$$

In Class Exercise

- Consider the recurrence $f(n) = 2f(n/2) + 1$, $f(1) = 1$
- Guess that $f(n) \leq cn - 1$:
- Q1: Show the base case - for what values of c does it hold?
- Q2: What is the inductive hypothesis?
- Q3: Show the inductive step.

Graph Induction: Coloring Graphs

- A *proper coloring* of a graph is an assignment of a color to each vertex such that every edge in the graph has two different colors at its endpoints.
- The *maximum* degree of a graph is maximum degree - number of neighbors - of any vertex.
- We can show that any graph with maximum degree 3 can be properly colored with at most 4 colors.

Induction

Fact: Any graph with maximum degree 3 can be properly colored with at most 4 colors. Proof by induction on n:

- BC: $n = 1$, a graph with 1 node can be colored with just 1 color
- IH: Any graph with $j < n$ nodes and maximum degree 3 can be colored with 4 colors
- IS: Consider any graph, G with n nodes and maximum degree at most 3. Remove any node v and its edges to get a graph G' that has $n - 1$ nodes and maximum degree at most 3. By the IH, we can color G' with at most 4 colors. Also, v has at most 3 neighbors in G'. Hence, we can assign v one of the 4 colors that does not appear on any of the 3 neighbors. This gives a proper coloring of G.

A Warning

- Warning: A common mistake is "build-up" induction (see next slides)
- A student *adds to* a structure for which the IH is assumed to be correct
- But this is done in a way that the IS does not hold for every way of creating the structure
- The best way to avoid this mistake is to start with an arbitrary structure of size n, and the apply the IH to a *smaller* problem

WRONG: “Build-up” Induction

Recall: A graph is *connected* if there is a path between every pair of nodes.
Claim: Any graph where every node has degree at least 2 is connected. “Proof” by induction on n.

- BC: $n = 3$, a triangle is connected
- IH: For all $j < n$, any graph with j nodes where each node has degree at least 2 is connected.
- IS: Consider some graph of size $n - 1$ with degree of every node equal to 2. By the IH, it is connected. Now, **add a node and two edges from that new node to the graph**. This new graph of size n is connected.

WRONG: "Build-up" Induction

- This "proof" is wrong! In fact, the claim is wrong - Can you find a counterexample?
- What happened? Build up does not ensure you're proving things for every required graph
- "Build-up" induction lures you into a tangled web of lies. Don't use it!
- Instead use "take away" induction: start with an arbitrary graph of the proper form, and then make it smaller in order to use the IH.
- "Take Away" induction is trustworthy. It doesn't work when you try to prove false things!

"Take-away" Induction Attempt

Claim: Any graph where every node has degree at least 2 is connected. Proof attempt by induction on n.

- BC: $n = 3$, a triangle is connected
- IH: For all $j < n$, any graph with j nodes where each node has degree at least 2 is connected.
- IS: Consider **an arbitrary graph, G with n nodes, each of which has degree at least** 2. Now, remove some node v and the edges that touch it from the graph G to get a new graph G'. Can we apply the IH to G'? No! Because some nodes in G' may not have degree at least 2, since their edges to v were removed.

So the proof fails, as it should, since the claim is false!

Also: “Smaller” isn’t always -1

- The IH only applies to smaller problems, but smaller doesn’t have to mean exactly 1 less.
- You’re unnecessarily restricting yourself if you assume that and there will be many (true) things you won’t be able to prove
- In the following proof, the subtrees T_1 and T_2 can range in size from $n-1$ all the way down to 1.

Inductive Proof

Fact: In any binary tree, the number of nodes with two children is one less than the number of leaves. Proof by induction on n:

BC: $n = 1$, there is 1 leaf node and 0 nodes with 2 children.
IH: $\forall j, 1 \leq j < n$, A binary tree with j nodes has a number of nodes with 2 children that is 1 less than the number of leaves.
IS: Consider an arbitrary binary tree, T with $n > 1$ nodes. If the root node has 1 child, let T_1 be the subtree rooted at that child, applying the IH to that subtree gives the result since the root node is neither a leaf nor a node with 2 children. If the root node has 2 children, let T_1 and T_2 be the subtrees rooted at each child and x_1, y_1, x_2, y_2 be the number of degree 2 nodes and leaves in each of them. By the IH, T_1 has $x_1 = y_1 - 1$ and T_2 has $x_2 = y_2 - 1$. Let x, y be the number of degree 2 nodes and leaf nodes in T. Then $x = x_1 + x_2 + 1 = (y_1 - 1) + (y_2 - 1) + 1 = y - 1$.

Reading

- "Proof by Induction" notes by Jeff Erickson (on class web page)
- Chapter 3 and 4, and Appendices in the text