1 Basic Complexity Theory

Definition: **P** = the set of all languages L that have a deterministic worst case polynomial time algorithm, A such that for all $x \in \Sigma^*$:

- if $x \in L$, $A(x)$ accepts
- if $x \in \bar{L}$, $A(x)$ rejects

Definition: **NP** = set of all languages L that have a deterministic worst case polynomial time algorithm, A, such that for all $x \in \Sigma^*$:

- if $x \in L$, there exists a string y such that $A(x, y)$ accepts and $|y|$ is bounded by a polynomial in $|x|$
- if $x \in \bar{L}$, for any string $y \in \Sigma^*$, $A(x, y)$ rejects

Definition For any complexity class C, we define $\mathbf{Co} - C = \{L|\bar{L} \in C\}$

1.1 co-C Examples

Definition: **co-P** = the set of all languages L such that there exists a deterministic algorithm, A running in worst case polynomial time such that for all $x \in \Sigma^*$:

- if $x \in \bar{L}$, $A(x)$ accepts
- if $x \in L$, $A(x)$ rejects

Note that $P = co - P$

Definition: **co-NP** = the set of all languages L such that there exists a deterministic algorithm, A running in worst case polynomial time such that for all $x \in \Sigma^*$:

- if $x \in \bar{L}$, there exists a string y such that $A(x, y)$ accepts and $|y|$ is bounded by a polynomial in $|x|$
- if $x \in L$, for any string $y \in \Sigma^*$, $A(x, y)$ rejects

1.2 Language Examples

- Let L_1 include only those strings (G, x) where the minimum cut size of G is x
- Let L_2 include only those strings (G, x) where G has a TSP tour of length x
- Let L_3 include only those strings (G, x) where G has no TSP tour of length x

- $L_1 \in P$
- $L_2 \in NP$
- $L_3 \in co - NP$

Questions:

- $L_1 \in NP$?
- $L_1 \in co - NP$?
- $L_2 \in P$?
- $L_3 \in P$?

2 Randomized Complexity Theory

2.1 RP and ZPP

Definition: **RP** = the set of all languages L that have a worst case polynomial time randomized algorithm, A, such that for all $x \in \Sigma^*$:

- if $x \in L$, $A(x)$ accepts with probability at least $1/2$
- if $x \in \bar{L}$, $A(x)$ rejects

Definition: **co-RP** = the set of all languages L that have a worst case polynomial time randomized algorithm, A, such that for all $x \in \Sigma^*$:

- if $x \in \bar{L}$, $A(x)$ accepts with probability at least $1/2$
- if $x \in L$, $A(x)$ rejects

Definition: **ZPP** = the set of all languages L that have an expected polynomial time algorithm, A, such that for all $x \in \Sigma^*$:

- if $x \in \bar{L}$, $A(x)$ accepts
- if $x \in L$, $A(x)$ rejects

Question: Does ZPP = co-ZPP?

Claim: $ZPP = \text{RP} \bigcap \text{co-RP}$

Question: How to show this?

2.2 BPP

Definition: **PP** = the set of all languages L that have a worst case randomized polynomial time algorithm, A, such that for all $x \in \Sigma^*$:

- if $x \in L$, $A(x)$ accepts with probability greater than $1/2$
- if $x \in \bar{L}$, $A(x)$ accepts with probability less than $1/2$

Note: this class is weak: we can not easily reduce the error probability with repetitions of the algorithm. The following class is more useful.

Definition: **BPP** = the set of all languages L that have a worst case randomized polynomial time algorithm, A, such that for all $x \in \Sigma^*$:

- if $x \in L$, $A(x)$ accepts with probability at least $3/4$
- if $x \in \bar{L}$, $A(x)$ accepts with probability no more than $1/4$