CS 591-06, Lecture 4

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Class Outline

- Feedback and Homeworks
- Review of “Augmenting Chains”
- Start of “Attack-Resistant P2P networks” talk

Feedback

- “More proofs”, “More examples” - will do
- “too much notation” - difficult to reduce amount of notation in a mathematical course. I’ll try to make notation more intuitive by giving more examples and more doing in-class exercises.
- “Leave out simple proofs” - will do, I might put these in hws.
- “Large intuitive leaps in proofs” - I’ll try to address this with more intuition about proofs

Review

“Marriage is not a word, it is a sentence” - King Vigor

- Hall’s Marriage Theorem gives us conditions under which there is a perfect matching but not an algorithm for finding it.
- The Find-Matching algorithm will find the largest possible matching
- The Find-Matching algorithm uses the idea of augmented chains.

Matching Algorithm

For a bipartite graph G with edge set E:

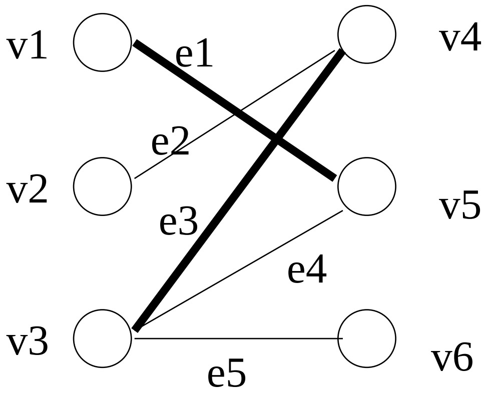
Defn: A *chain* is a sequence of vertices and edges $v_1, e_1, v_2, e_2, ..., e_l, v_l$ such that the edge $e_i = (v_{i-1}, v_i)$.

Let M be some (possibly non-maximum) matching on G.

Defn: A chain $v_1, e_1, v_2, e_2, \ldots, e_l, v_l$ is called *M-alternating* if the edges $e_2, e_4, \ldots$ are in M.

Example

$M = \{e1, e3\}$
$C = \{v6, e5, v3, e3, v4\}$ is an M-alternating chain
$C' = \{v6, e5, v3, e4, v5\}$ is not an M-alternating chain



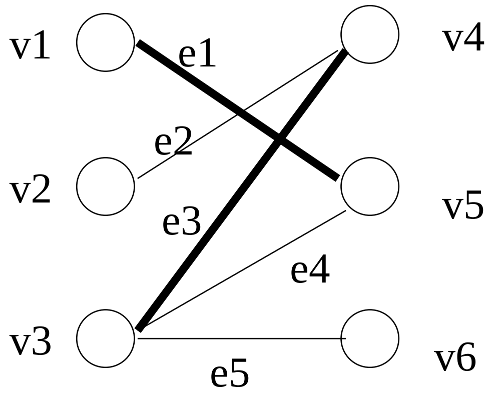
M-Augmenting Chain

Defn: A chain $v_1, e_1, v_2, e_2, \ldots, e_l, v_l$ is called *M-augmenting* if it is M-alternating and the vertices v_1 and v_l are not incident to any edges in M.

Chain Theorem: A Matching M is maximum for a graph G if and only if G contains no M-augmenting chains.

Example

$M = \{e1, e3\}$
$C = \{v6, e5, v3, e3, v4, e2, v2\}$ is an M-augmenting chain
$C' = \{v6, e5, v3, e4, v5\}$ is not an M-augmenting chain



Algorithm Find-Matching

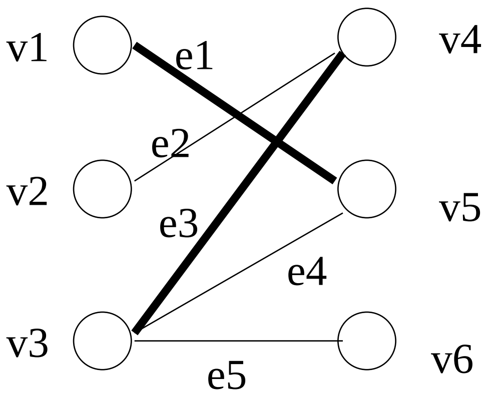
Input: A bipartite graph $G = (L, R)$

Algorithm *Find-Matching*:

1. Let $M = \phi$
2. While there exist M-augmenting chains do:
   (a) Let $A = v_1, e_1, v_2, e_2, v_3, e_3 \ldots, e_{l-1}, v_{l-1}, e_l, v_l$ be some M-augmenting chain.
   (b) Set the new matching M to be $M \cup \{e_1, e_3, \ldots, e_l\} - \{e_2, e_4, \ldots, e_{l-1}\}$

Exercise

In the graph below, let $M = \{e1, e3\}$
Increase the size of M by finding an M-augmenting chain.
Give the M-augmenting chain and the new larger matching.



Algorithm Find-Matching

Input: A bipartite graph $G = (L, R)$

Algorithm *Find-Matching*:

1. Let $M = \phi$
2. While there are still unmatched vertices in L do
   (a) For each v that is unmatched in L do
      i. Search for an M-augmenting chain by doing a breadth first search from v
      ii. If a chain was found, let $A = v_1, e_1, v_2, e_2, v_3, e_3 \ldots, e_{l-1}, v_{l-1}, e_l, v_l$ be that chain. Set the new matching M to be $M \cup \{e_1, e_3, \ldots, e_l\} - \{e_2, e_4, \ldots, e_{l-1}\}$

Proof of Chain Theorem

Chain Theorem: A Matching M is maximum for a graph G if and only if G contains no M-augmenting chains.

Easy direction: If a matching M is maximum then G contains no M-augmenting chain.

To show this, assume, by way of contradiction, that M is a maximum matching and that there is an M-augmenting chain $C = v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, \ldots, e_l, v_l$. Let $E' = M \cup \{e_1, e_3, \ldots, e_l\} - \{e_2, e_4, \ldots, e_{l-1}\}$.

Easy direction

We note the following:

- $|E'| > |M|$
- The edge set E' is a matching on G.

This implies that E' is a larger matching than M which is a contradiction.

Hard Direction

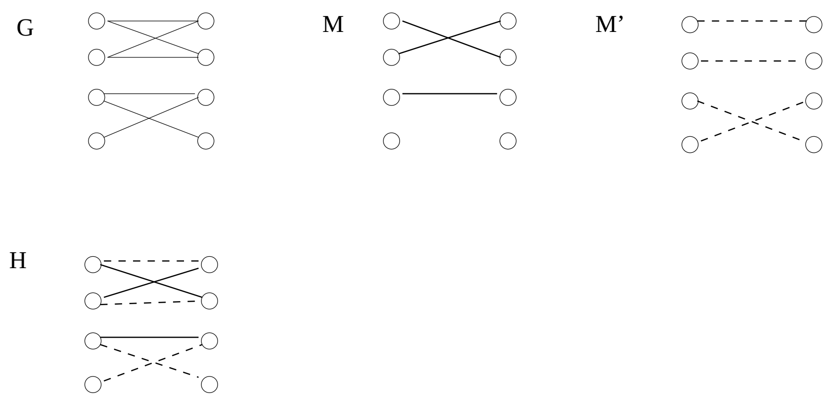
If G contains no M-augmenting chain, then the matching M is maximum.

Proof: Assume there exists some matching M' such that $|M'| > |M|$.

Let H be the subgraph of G which has only those edges which appear in either M or M' but not both. We know:

- H is a graph of degree no more than 2 (it consists of cycles and paths)
- The edges of each cycle or path in H alternate between being in M and M'.

Picture of H



More about H

- All cycles in H are of even length.
- There must be some path in H with more M' edges than M edges.
- This path is an M-augmenting chain

Experimental Results