CS 591-06, Lecture 9

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Class Outline

A reasonable probability is the only certainty. - E.W. Howe

- Review
- Expander Graph Proof
- In Class Exercise
- Intro to Chernoff Bounds

Union Bound

Union Bound Fact: Let Γ be any event space, P be any probability function and $\gamma_1, \gamma_2, \ldots \gamma_n$ be any events in Γ. Then:

$$P(\bigcup_{i=1}^{n} \gamma_i) \leq \sum_{i=1}^{n} P(\gamma_i)$$

Example:

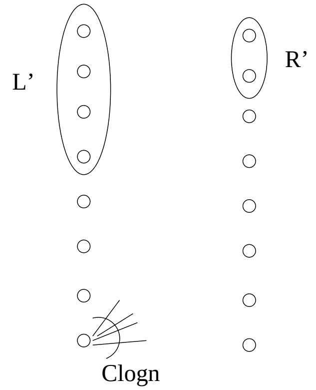
Let γ_1 be the event that coin 1 is heads, γ_2 be the event that coin 2 is heads. Then:

$$P(\gamma_1 \bigcup \gamma_2) \leq P(\gamma_1) + P(\gamma_2)$$

Expander Graphs

- Assume we've got a biparitite graph $G = (L, R)$, where each vertex in L has $C \log n$ edges to random nodes in R. (The edges are chosen uniformly and independently at random),
- Consider some fixed sets $L' \subset L, R' \subset R$ such that $|L'| = n/2$ and $|R'| = \epsilon n / \log n$, for some fixed $\epsilon > 0$
- **Question:** What is the probability that there is no edge between the set L' and the set R'?

Expander Picture



Probability of Failure

- Q: What is probability that a single edge from L' does not land in R'?
- Q: What is the probability that *no* edge from L' lands in R'?
- Note: This probability is the same no matter which fixed set L' and R' are chosen such that L' and R' are of the appropriate size.

Probability of Failure

- Q: What is probability that a single edge from L' does not land in R'?
- A: $1 - \epsilon / \log n$
- Q: What is the probability that *no* edge from L' lands in R'?
- A: Since there are $n/2 * C \log n$ independent edges, it is:

$$(1 - (\epsilon / \log n))^{n/2 * C \log n} \leq e^{-\frac{\epsilon C n}{2}}.$$

- This probability is the same no matter which fixed set L' and R' are chosen such that L' and R' are of the appropriate size.

Union Bound

What is the probability that any of the many, many possible sets of appropriate size fail?

Q: How many sets of size $n/2$ are there in L?

Q: How many sets of size $\epsilon n / \log n$ are there?

Q: What is an upper bound on the probability of *any failure*?

Union Bound

What is the probability that any of the many, many possible sets of appropriate size fail?

Q: How many sets of size $n/2$ are there in L? A: $\binom{n}{n/2}$

Q: How many sets of size $\epsilon n/\log n$ are there? A: $\binom{n}{\epsilon n/\log n}$

Q: What is an upper bound on the probability of *any failure*? A:

$$\binom{n}{n/2} * \binom{n}{\epsilon n/\log n} * (\epsilon/\log n)$$

Some Useful Bounds

- Stirlings Formula: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + O(\frac{1}{n^2})\right)$
- $n! \geq \left(\frac{n}{e}\right)^n$
- $\binom{n}{k} \leq \frac{n^k}{k!}$
- $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$

An upper bound

Q: What is an upper bound on the probability of *any failure*? A:

$$\begin{aligned}
& \binom{n}{n/2} * \binom{n}{\epsilon n/\log n} * (e^{\frac{-\epsilon C n}{2}}) && (1)\\
\leq\ & \left(\frac{ne}{n/2}\right)^{n/2} * \left(\frac{n}{\epsilon n/\log n}\right)^{\epsilon n/\log n} * (e^{\frac{-\epsilon C n}{2}}) && (2)\\
=\ & (2e)^{n/2} * (2e)^{n/2} * (e^{\frac{-\epsilon C n}{2}}) && (3)\\
=\ & (2e)^{n} * (e^{\frac{-\epsilon C n}{2}}) && (4)\\
\leq\ & e^{2n - \frac{\epsilon C n}{2}} && (5)\\
\leq\ & e^{n(2 - \frac{\epsilon C}{2})} && (6)
\end{aligned}$$

An Upper Bound

- For $C > 4/\epsilon$, this quantity can be made *exponentially* small in the size of n
- This implies that for C chosen as a function of ϵ only, we can be sure that a random bipartite graph with $C \log n$ edges from each vertex on the left side has "expansion" properties with high probability.
- The probability of failure gets smaller and smaller as n grows large

In Class Exercise

- Consider a simple random general graph G on n vertices where each of the $\binom{n}{2}$ possible edges is included with probability p.
- Q: What is an upperbound on the probability that the graph contains no independent set of size $n/2$?
  - Q1: What is the probability that some fixed set of $n/2$ vertices is an independent set? (i.e. has no edges)
  - Q2: What is the probability that *no* set of $n/2$ vertices is an independent set?

Chernoff Bounds

- Let $X = X_1, \ldots, X_n$ where the X_i are random variables which are: independent and all take on values between 0 and 1.

then for any $\lambda > 0$,

- $P(|X - E(X)| \leq \lambda) \leq 2e^{\frac{-\lambda^2}{2m}}$

Chernoff Example

- Let $X = X_1, \ldots, X_n$, where each X_i is 1 if the i-th flip of a coin flip is heads tails otherwise. Let the coin turn up heads with probability 1/2.
- Q: What is the probability that the number of heads is $(3/4)m$?